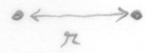
Lecture 1

$\epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{Nm^2}$

⊙ Force between two charges at rest



$$F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}$$

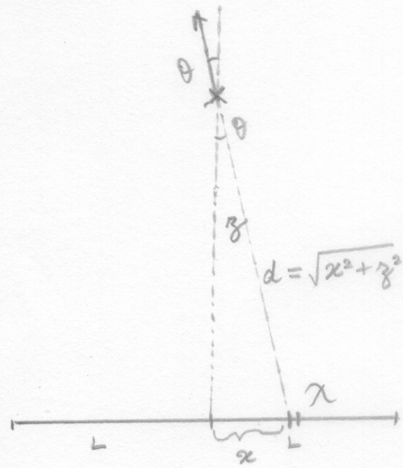
We break this down into an electric field produced by a charge and another charge feeling the force.

$$\vec{F}_1 = q_1 \vec{E} \quad \& \quad \vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q_2 \hat{r}}{r^2}$$ where $\hat{r}$ is from q_2 to q_1

Does this breakdown add any insight? No. Not yet.

⊙ Explain the principle of superposition the $\vec{E}$ field.

Example 1. $\vec{E}$ field due to an finite line charge.



Field due to dx: $$dE_z = \frac{1}{4\pi\epsilon_0} \frac{\lambda dx}{(x^2+z^2)} \cos\theta$$

But, $$\cos\theta = \frac{z}{\sqrt{x^2+z^2}}$$

$$\therefore dE_z = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda dx \, z}{(x^2+z^2)^{3/2}}$$

$$\therefore E_z = \frac{\lambda z}{4\pi\epsilon_0} \int_{-L}^{L} \frac{dx}{(x^2+z^2)^{3/2}}$$

$$\therefore E_z = \frac{\lambda \cancel{z}}{4\pi\epsilon_0} \int_{\theta_1}^{\theta_2} \frac{\cancel{z} \sec^2\theta \, d\theta}{\cancel{z} z^{\cancel{3}} (1+\tan^2\theta)^{3/2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0 z} \int_{\theta_1}^{\theta_2} \cos\theta \, d\theta = \frac{\lambda}{4\pi\epsilon_0 z} [\sin\theta_2 - \sin\theta_1]$$

$$\boxed{E_z = \frac{2\lambda}{4\pi\epsilon_0 z} \frac{L}{\sqrt{L^2+z^2}}} \quad \boxed{E_x = E_y = 0}$$

$t^2 = x^2 + z^2$

$\cancel{2} t \, dt = \cancel{2} x \, dx$

$x = z \tan\theta$

$\therefore dx = z \sec^2\theta \, d\theta$

$1 + \tan^2\theta$

$= \sec^2\theta$

Now, take $L \to \infty$

we get: $$E_z = \frac{2\lambda}{4\pi\epsilon_0 z} = \frac{\lambda}{2\pi\epsilon_0 z}.$$ Does this make sense?

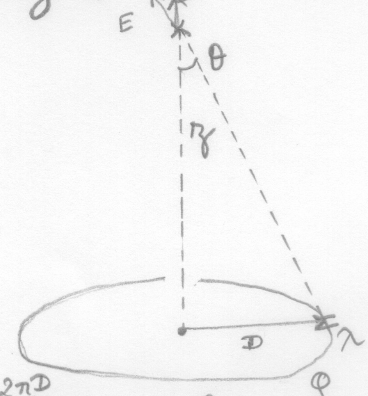
And, take ~~$z \to \infty$~~; $z \gg L$

we get: $$E_z = \frac{2\lambda}{4\pi\epsilon_0 z^2}\frac{L}{\sqrt{\frac{L^2}{z^2}+1}} \to 1$$

$$E_z = \frac{q}{4\pi\epsilon_0 z^2}$$ (a point charge!)

Example 2

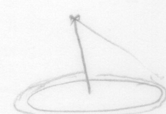
$\vec{E}$ field due to a ring of charge



$$E_z = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi D}\frac{\lambda\, dl}{z^2+D^2}\cos\theta$$

$$= \frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda . 2\pi D}{(z^2+D^2)^{3/2}}\cdot z = \boxed{\frac{Q}{4\pi\epsilon_0}\frac{z}{(z^2+D^2)^{3/2}}}$$

$$E_z = \frac{1}{2\epsilon_0}\frac{\lambda z D}{(z^2+D^2)^{3/2}}$$



Extend to a disc of charge of radius $= D$.; $\sigma = Q/\pi D^2$

$$t^2 = z^2 + r^2$$
$$dt = 2r dr$$
$$\int t^{-3/2}\frac{dt}{2}$$

$$E_z = \frac{z}{4\pi\epsilon_0}\int_0^D \frac{2\pi r dr\, \sigma}{(z^2+r^2)^{3/2}}$$

$$= \frac{z\sigma}{2\epsilon_0}\int_0^D \frac{r dr}{(z^2+r^2)^{3/2}}$$

$$= \frac{z\sigma}{4\epsilon_0}\int_{z^2}^{z^2+D^2} t^{-3/2} dt$$

$$= -\frac{z\sigma}{\cancel{2}\cancel{4}\epsilon_0}\cancel{2}\left[t^{-1/2}\right]_{z^2}^{z^2+D^2} = \frac{z\sigma}{2\epsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{z^2+D^2}}\right]$$

$$\boxed{E_z = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+D^2}}\right]}$$

Taking the limit of $D \to \infty$,

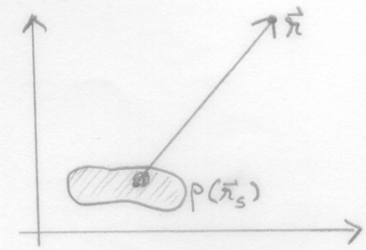
$$\boxed{E_z = \frac{\sigma}{2\epsilon_0}} \leftarrow \text{field near an infinite plane of charge.}$$

Same if $z \to 0$.

Also note that in going across the plane, the E field <u>changes</u> by "$\frac{\sigma}{\epsilon_0}$". Very interesting!

Going from Coulomb's Law (Green's function form) to differential form.

What is the $\vec{E}$ field produced at $\vec{r}$ due to a charge distribution $\rho(\vec{r}_s)$?



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}_s)\, dV}{(\vec{r} - \vec{r}_s)^3} (\vec{r} - \vec{r}_s)$$

We will try to find the divergence of $\vec{E}(\vec{r})$, i.e. $\vec{\nabla}_r \cdot \vec{E}(\vec{r})$

So, we apply it,

$$\vec{\nabla}_r \cdot \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{r}_s)\, dV \; \vec{\nabla} \cdot \left(\frac{\vec{r} - \vec{r}_s}{(\vec{r} - \vec{r}_s)^3} \right)$$

But, $\boxed{\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) = 4\pi \delta^3(\vec{r})}$

$$\text{So, } \vec{\nabla}_r \cdot \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{r}_s)\, dV \; 4\pi \delta^3(\vec{r} - \vec{r}_s)$$

$$\boxed{\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}} \leftarrow \text{First of Maxwell's Laws}$$

Applying the curl to $\vec{E}$ would give us $\vec{\nabla} \times \vec{E} = 0$.

Only true in the absence of changing magnetic fields.

What are the other laws you know?

<u>The potential</u>

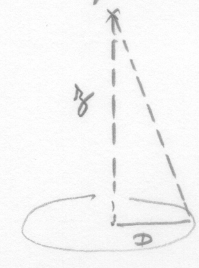
Notice that $\boxed{\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}}$ —— Prove it in H.W.

$$\therefore \vec{E}(\vec{r}) = \frac{-1}{4\pi\epsilon_0} \int_V \rho(\vec{r}_s)\, dV \; \vec{\nabla}_r \left(\frac{1}{|\vec{r} - \vec{r}_s|} \right)$$

$$= -\vec{\nabla}_r \left(\frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}_s)}{|\vec{r} - \vec{r}_s|} dV \right) \leftarrow \text{Potential } \phi(r) \,; \; \boxed{\vec{E} = -\vec{\nabla}\phi}$$

Example

Can you solve the ring problem with potential?



$$V = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{z^2+D^2}}$$

$$\therefore\ E_z = -\frac{\partial V}{\partial z} = \frac{q}{4\pi\epsilon_0}\frac{z}{(z^2+D^2)^{3/2}}$$

Gauss' Law Examples.

$$\boxed{\int_V \vec{\nabla}\cdot\vec{E}\, dV = \oint_S \vec{E}\cdot d\vec{A}}$$

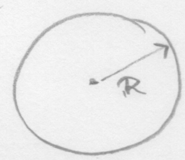
$$\boxed{\vec{\nabla}\cdot(\vec{\nabla}\phi) = -\rho/\epsilon_0}$$

Poisson Eqn: $\nabla^2\phi = -\rho/\epsilon_0$

Laplace Eqn: $\nabla^2\phi = 0$

⊙ Spherical Symmetry

Shell of charge q



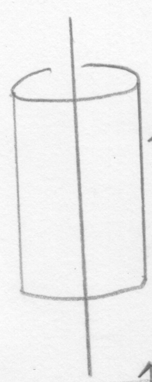
Find $\vec{E}$ & ϕ inside and outside.

Outside: $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$; $\phi = \int_\infty^R \vec{E}\cdot d\vec{x} = \frac{q}{4\pi\epsilon_0 r}$

Inside: $\vec{E} = 0$; $\phi = \frac{q}{4\pi\epsilon_0 R}$

ϕ is continuous across a surface of charge!

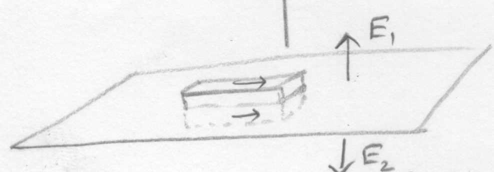
⊙ Cylindrical Symmetry



$$2\pi r \ell E = \frac{\lambda \ell}{\epsilon_0}$$

$$\therefore\ \boxed{E = \frac{\lambda}{2\pi\epsilon_0 r}}$$

⊙



$$E_z = \frac{\sigma}{2\epsilon_0}$$

~~Continuity~~ Matching conditions: $E_{z1} - E_{z2} = \frac{\sigma}{\epsilon_0}$

$E_{x1} = E_{x2}$ from the curl.