Uniqueness Theorems — Lecture 10

Laplace's Eqn by itself does not determine Φ. You need boundary conditions.

What are the boundary conditions needed for this?

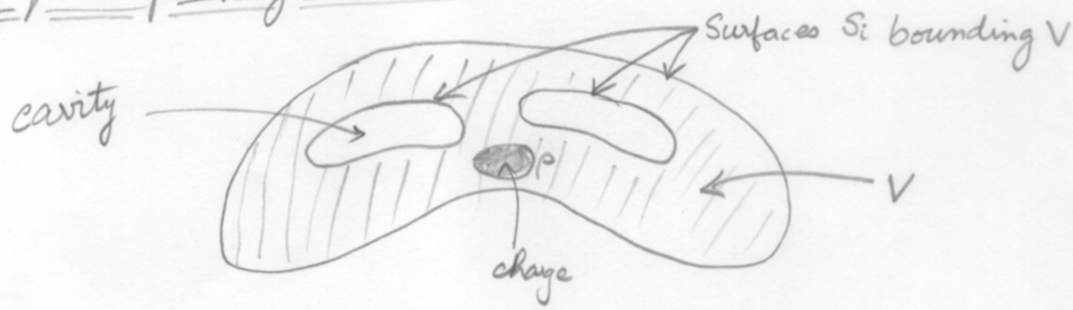
1-D is easy to understand.

2-D and 3D is not trivial.

If, given boundary conditions, you find <u>one</u> valid solution to Φ (even by guessing) then it is the <u>only</u> valid solution for Φ.

- Intuition would suggest Φ at boundary would uniquely specify Φ inside
- But what if we know $\vec{E}$ (i.e. $-\vec{\nabla}\Phi$) at boundary. Can we know Φ inside?
- And what about Φ on some part of the boundary and $\vec{E}$ elsewhere?

Uniqueness proof by contradiction



Suppose V supports two solutions to $\nabla^2\Phi = \rho/\epsilon_0$

Let's call these solutions Φ_1, Φ_2

Then, define $\eta = \Phi_1 - \Phi_2$; $\quad \nabla^2\Phi_1 = \rho/\epsilon_0$, $\nabla^2\Phi_2 = \rho/\epsilon_0$

Then, $\nabla^2\eta = 0$ should be true inside.

Using Green's Identity,

$$\vec{\nabla}\cdot(f\vec{\nabla}g) = (\vec{\nabla}f)\cdot(\vec{\nabla}g) + f\nabla^2 g$$

$$\therefore \int\vec{\nabla}\cdot(f\vec{\nabla}g)\,dV = \int(\vec{\nabla}f)\cdot(\vec{\nabla}g)\,dV + \int f\nabla^2 g\,dV$$

Replace both $f = g = \eta$

So,

$$\oint \eta\vec{\nabla}\eta\cdot d\vec{A} = \int|\vec{\nabla}\eta|^2\,dV + \int\eta\underbrace{\nabla^2\eta}_{\to\, 0 \text{ everywhere.}}\,dV$$

So,

$$\oint \eta\vec{\nabla}\eta\cdot d\vec{A} = \int|\vec{\nabla}\eta|^2\,dV$$

Now, $\int |\vec{\nabla}\eta|^2 dV$ is non-negative.

$\hookrightarrow = 0$ if

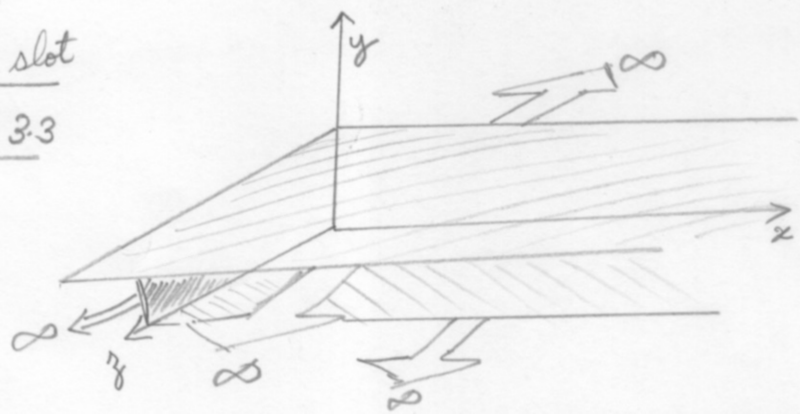
- $\eta = 0$ on boundaries ; Dirichlet boundary conditions.
  $\Phi_1(\vec{r}_s) = \Phi_2(\vec{r}_s) \rightarrow \eta(\vec{r}) = 0$
- or, $\vec{\nabla}\eta = 0$
  $\therefore$ $E_1 = E_2$ on boundary , Neumann boundary condition
  $\hookrightarrow \eta(\vec{r}) = 0$
- or, $\eta = 0$ in some parts of S, $\vec{\nabla}\eta$ in other parts of S
  No overlaps. Mixed boundary conditions

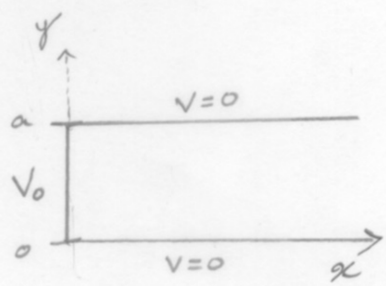
Finding the potential by directly solving the Laplace Equation for a region with no charge.

$$\nabla^2 \Phi = 0$$

An infinite slot

Griffiths Example 3.3





Two infinite grounded metal plates lie parallel to the xz plane, one at $y=0$, and the other at $y=a$. The left end at $x=0$ is closed off with an infinite strip that is insulated from the two plates and kept at constant voltage $V_0(y)$. Find the potential inside the slot

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \quad \longleftarrow \text{Essentially a 2D problem.}$$

We have Dirichlet boundary conditions.

i) $\Phi = 0$ at $y = 0$

ii) $\Phi = 0$ at $y = a$

iii) $\Phi = V_0(y)$ at $x = 0$ between $0 < y < a$

iv) $\Phi = 0$ at $x \rightarrow \infty$

Look for solutions of the form

$$\Phi(x,y) = X(x)\,Y(y)$$

to be able to separate solutions. Uniqueness tells us *if* we can find a solution, then that is the only and correct solution.

So plugging in that assumed solution:

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{c_1} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{c_2} = 0$$

The two ODEs must equal constants, else they would not be separable.

And the constants must add up to zero $c_1 + c_2 = 0$

Take c_1 to be +ve, and $c_2 = -c_1$ to be negative

$$\frac{1}{X}\frac{d^2X}{dx^2} = c_1 \quad ; \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -c_1$$

$c_1 = k^2$ (positive)

$$\frac{d^2X}{dx^2} = k^2X \longleftarrow \text{has exponential solutions}$$

$$\frac{d^2Y}{dy^2} = -k^2Y \longleftarrow \text{has oscillatory solutions.}$$

You can build any function out of oscillatory functions, even piecewise discontinuous ones.

But not always out of exponentials. They best model decays.

So, $X(x) = \underbrace{Ae^{kx}}_{0 \text{ because of (iv)}} + Be^{-kx}$

$$Y(y) = C\sin ky + \underbrace{D\cos ky}_{0 \text{ because of (i)}}$$

So, generally, $\Phi(x,y) = X(x)Y(y) = Ce^{-kx}\sin ky$

Using (ii), we can find "quantum" of k

$$\sin(ky) = 0 \text{ at } y = a$$

So, $$ka = n\pi$$

$$k = \frac{n\pi}{a}$$

So, $$\boxed{\Phi(x,y) = Ce^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)}$$

At $x=0$, $\Phi(0,y) = C\sin\left(\frac{n\pi y}{a}\right)$

does not necessarily match the imposed form $\Phi(0,y) = V_0(y)$ [iii]

But that is not a problem because we can add up multiple solutions linearly to match $V_0(y)$ at the "hot" surface.

So, $$\Phi(0,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$$

One can determine C_n by exploiting orthogonality & completeness of $\sin(x)$.

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \int_0^a V_0(y) \sin\left(\frac{n'\pi y}{a}\right) dy$$

Only integrals where $n = n'$ survive.

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \frac{a}{2} \text{ if } n = n'$$

Therefore, $$C_n = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right) dy$$

Once we have these coefficients, we can plug them back into the general solution.

Let's solve for C_n for $V_0(y) = V_0$ .. a constant

$$C_n = \frac{2V_0}{a}\int_0^a \sin\left(\frac{n\pi y}{a}\right) dy$$

$$= \frac{2V_0}{\frac{n\pi}{\cancel{a}}\cancel{a}}\left[\cos 0 - \cos(n\pi)\right]$$

$C_n = \frac{4V_0}{n\pi}$ for $n = 1, 3, 5, \ldots$ odd, Zero otherwise.

So, the solution would look like

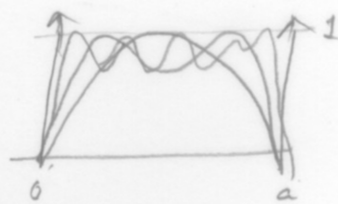
$$\boxed{\Phi(x,y) = \sum_{n=1,3,5}^{\infty} \frac{4V_0}{n\pi} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)}$$

for the hot strip at constant voltage V_0.

Plot this on a computer!

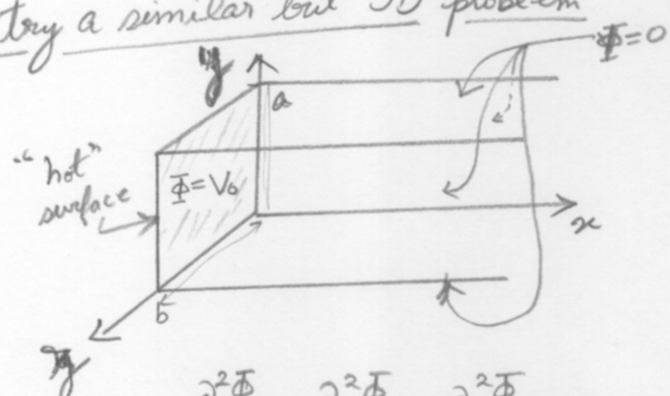
In the y-dimension, till it approximates V_0.

Be wary of the edges! "overshoot"



Each y mode has its own rate of decay in x.

Now, let us try a similar but 3D problem



Conditions

i) $\phi = 0$ at $y = 0$

ii) $\phi = 0$ at $y = a$

iii) $\phi = 0$ at $z = 0$

iv) $\phi = 0$ at $z = b$

v) $\phi \to 0$ at $x \to \infty$

vi) $\phi = V_0$ at $x = 0$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Assuming $\Phi(x,y,z) = X(x)\,Y(y)\,Z(z)$,

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\substack{+\text{ve } c_1 \\ \text{decaying} \\ = k^2+l^2}} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\substack{-\text{ve } c_2 \\ \text{oscillatory} \\ = -k^2}} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{\substack{-\text{ve } c_3 \\ = -l^2}} = 0$$

So, $\dfrac{d^2X}{dx^2} = (k^2+l^2)X \leftarrow$ exponential

$$\left.\begin{aligned} \frac{d^2Y}{dy^2} &= -k^2 Y \\ \frac{d^2Z}{dz^2} &= -l^2 Z \end{aligned}\right\} \text{oscillatory}$$

Solutions:

$$X = A e^{-\sqrt{k^2+\ell^2}\,x} \quad \longleftarrow \text{ we are keeping 'just the decaying term (v)}$$

$$Y = C \sin(ky) \quad \longleftarrow \text{ cosine term can't satisfy (i)}$$

$$Z = D \sin(\ell z) \quad \longleftarrow \text{ cosine term can't satisfy (iii)}$$

So, general solution:

$$\Phi(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\, e^{-\sqrt{\left(\frac{m}{b}\right)^2+\left(\frac{n}{a}\right)^2}\,x} \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)$$

$ka = n\pi$

$\therefore k = n\pi/a$

$\ell b = n\pi$

$\therefore k = n\pi/b$

We want to fit the boundary term at the hot plate.

$$V_0 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)$$

Using orthogonality, we can evaluate $C_{n,m}$.

$$V_0 \int_0^a \sin\left(\frac{n'\pi y}{a}\right) dy \int_0^b \sin\left(\frac{m'\pi z}{b}\right) dz = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \int_0^a \sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{n\pi y}{a}\right) dy \times \int_0^b \sin\left(\frac{m'\pi z}{b}\right)\sin\left(\frac{m\pi z}{b}\right) dz$$

$\frac{2}{n'\pi}$ for odd n' $\qquad \frac{2}{m'\pi}$ for odd m'

$$= C_{n',m'}\,\frac{a}{2}\,\frac{b}{2} \quad \text{(only } n=n',\ m=m' \text{ survive)}$$

$$\therefore \boxed{C_{n,m} = \frac{16 V_0}{nm\pi^2} \text{ for } n \,\&\, m \text{ are both odd}}$$

&

solution is therefore:

$$\Phi(x,y,z) = \frac{16 V_0}{\pi^2} \sum_{\substack{n=1,3,5\\ \text{odd}}}^{\infty}\ \sum_{\substack{m=1,3,5\\ \text{odd}}}^{\infty} \frac{1}{n\pi}\, e^{-\pi x\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}} \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{a}\right)$$

Consider how you would separate variables for spherical problems & cylindrical problems.