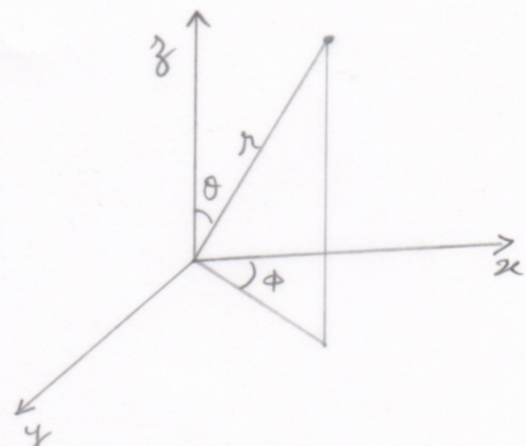
Lecture 11 — Laplace Equation in Spherical Coordinates.

Basics of spherical coordinates



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

So,

$$dx = \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi$$
$$dy = \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi$$
$$dz = \cos\theta\, dr - r\sin\theta\, d\phi$$

So, $ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$

Definitions of Gradient

$$\vec{\nabla}\Phi = \hat{r}\frac{\partial\Phi}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial\Phi}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}$$

Definition of Laplacian

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

So, let us try to solve $\nabla^2\Phi = 0$ in spherical coordinates, appropriate for problems with spherical symmetry.

ATTEMPT 1 Assume we have azimuthal symmetry. That is, $\frac{\partial\Phi}{\partial\phi} = 0$

Then, things get a bit separable.

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) = 0$$

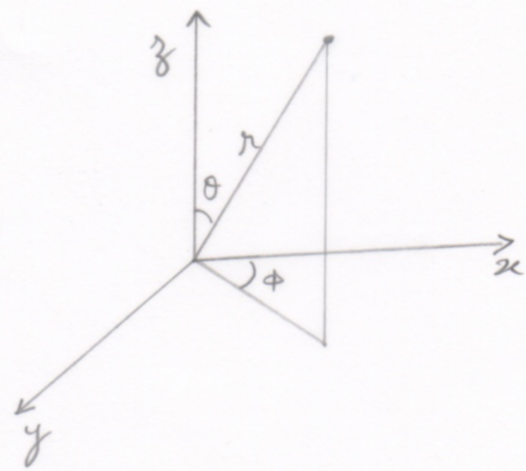
Assume $\Phi(r,\theta) = R(r)\,\eta(\theta)$

Plugging it in, we achieve separation of variables.

$$\eta(\theta)\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) + \frac{R(r)}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Phi}{d\phi}\right) = 0$$

Lecture 11

Laplace Equation in Spherical Coordinates.



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Dividing throughout by $\Phi = R\eta$

$$\underbrace{\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\ell(\ell+1)} + \underbrace{\frac{1}{\eta(\theta)\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\eta}{d\theta}\right)}_{-\ell(\ell+1)} = 0$$

So,

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \ell(\ell+1)R \longleftarrow \text{Polynomial solutions}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\eta}{d\theta}\right) = -\ell(\ell+1)\eta\sin\theta \longleftarrow \text{Legendre equation}$$

How?

$\hookrightarrow$ Let $x = \cos\theta$

$dx = -\sin\theta\, d\theta$

$d\theta = \frac{-dx}{\sin\theta}$

$$\cancel{\sin\theta}\frac{d}{dx}\left((1-x^2)\frac{d\eta}{dx}\right) = -\ell(\ell+1)\eta\cancel{\sin\theta}$$

$$\therefore \boxed{\frac{d}{dx}\left((1-x^2)\frac{d\eta}{dx}\right) = \ell(\ell+1)}$$

Legendre diff eq

Their solutions are:

$$\boxed{\begin{aligned} R(r) &= Ar^\ell + \frac{B}{r^{\ell+1}} \\ \eta(\theta) &= P_\ell(\cos\theta) \end{aligned}}$$

Put them together,

$$\boxed{\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(Ar^\ell + \frac{B}{r^{\ell+1}}\right)P_\ell(\cos\theta)}$$

We have worked on problems for this sort of solution for dipoles, polarization, conductors.

ATTEMPT 2 Solution in spherical coordinates with no azimuthal symmetry

Assume $\Phi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)$

$$Y\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{R}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = 0$$

So, $$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1) \longrightarrow \text{Polynomial solution.}$$

$$\frac{1}{Y\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -l(l+1) \longrightarrow \text{Spherical harmonics}$$

$Y_{l,m}(\theta,\phi)$ like this

m stays between $-l$ and l.

$$Y_{0,0}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

Sort of Legendre in θ

Oscillatory in ϕ

$$Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\sin\phi$$

INSIGHT Where do these spherical harmonics pop up?

Answer: Anywhere where $\frac{1}{|\vec{r}-\vec{r}'|}$ is involved. And $\frac{1}{|\vec{r}-\vec{r}'|}$ is Green's function solution to $\nabla^2 G = 4\pi\delta^3(\vec{r}-\vec{r}')$

How come? If we expand $\frac{1}{|x-x'|}$ around x;

$$\frac{1}{|x-x'|} = \frac{1}{x}\left(1-\frac{x'}{x}\right)^{-1} = \frac{1}{x}\left(1+\left(\frac{x'}{x}\right)+\left(\frac{x'}{x}\right)^2+\left(\frac{x'}{x}\right)^3\right]$$

A similar expansion can be done in 3D.

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\left(\frac{r'}{r}\right)^l\sum_{m=-l}^{l}Y^*_{lm}(\theta,\phi)\,Y_{lm}(\theta,\phi)$$

ATTEMPT 3 Can we tease apart the angular harmonics?

$$\frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -\ell(\ell+1)$$

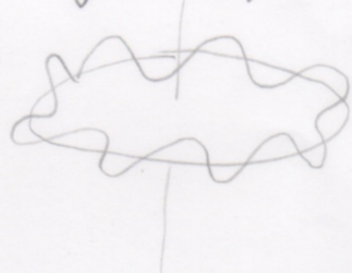
Let's assume $Y(\theta,\phi)=\eta(\theta)\chi(\phi)$

$$\chi\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\eta}{d\theta}\right)+\eta\frac{d^2\chi}{d\phi^2} = -\ell(\ell+1)\sin^2\theta\,\chi\eta$$

$$\underbrace{\frac{1}{\eta}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\eta}{d\theta}\right)+\ell(\ell+1)\sin^2\theta}_{+m^2} + \underbrace{\frac{1}{\chi}\frac{d^2\chi}{d\phi^2}}_{-m^2} = 0$$

$-m^2$ for oscillatory solutions.

Why? Like a de-Broglie atom!



$m\lambda = 2\pi r$

'situation'

$$\frac{d^2\chi}{d\phi^2} = -m^2\chi$$

$\hookrightarrow \chi = A\cos(m\phi)+B\sin(m\phi)$

And,

$$\frac{1}{\eta}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\eta}{d\theta}\right)+\ell(\ell+1)\sin^2\theta = m^2$$

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\eta}{d\theta}\right) = \left[m^2-\ell(\ell+1)\sin^2\theta\right]\eta$$

Let $x = \cos\theta$; $\sin\theta = 1-x^2$

$dx = -\sin\theta\, d\theta$

$d\theta = \frac{-dx}{\sin\theta}$

$$\cancel{+\sin^2\theta}\frac{d}{dx}\left(\sin^2\theta\frac{d\eta}{dx}\right) = \left[\frac{m^2}{\sin^2\theta}-\ell(\ell+1)\cancel{\sin^2\theta}\right]\eta$$

$$\frac{d}{dx}\left((1-x^2)\frac{d\eta}{dx}\right) = \left[\frac{m^2}{1-x^2}-\ell(\ell+1)\right]\eta$$ ← Associated Legendre Diff Eq.

$\eta(\theta) = P_\ell^m(\theta)$ → Associated Legendre polynomials.

Underlying structure of spherical harmonics $Y(\ell)$

So,

$$\boxed{Y(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} C_{\ell,m}P_\ell^m(\theta)\cos(m\phi)}$$