Lecture 12

Poisson's Equation

$$\nabla^2 \Phi = \rho/\epsilon_0$$

This is the fundamental theorem of electrostatics.

More fundamental than Coulomb's Law, which holds only when $\Phi(\infty) \to 0$.

Being able to solve Poisson's Equation is relevant when we have sources of charge, ρ, and boundary conditions on Φ or $\frac{\partial \Phi}{\partial \hat{n}}$.

In general, solutions is of the form:

$$\Phi(\vec{r}) = \underbrace{\frac{q}{4\pi\epsilon_0|\vec{r}-\vec{r}_0|}}_{\text{charge is here}} + \Phi_0(\vec{r})$$

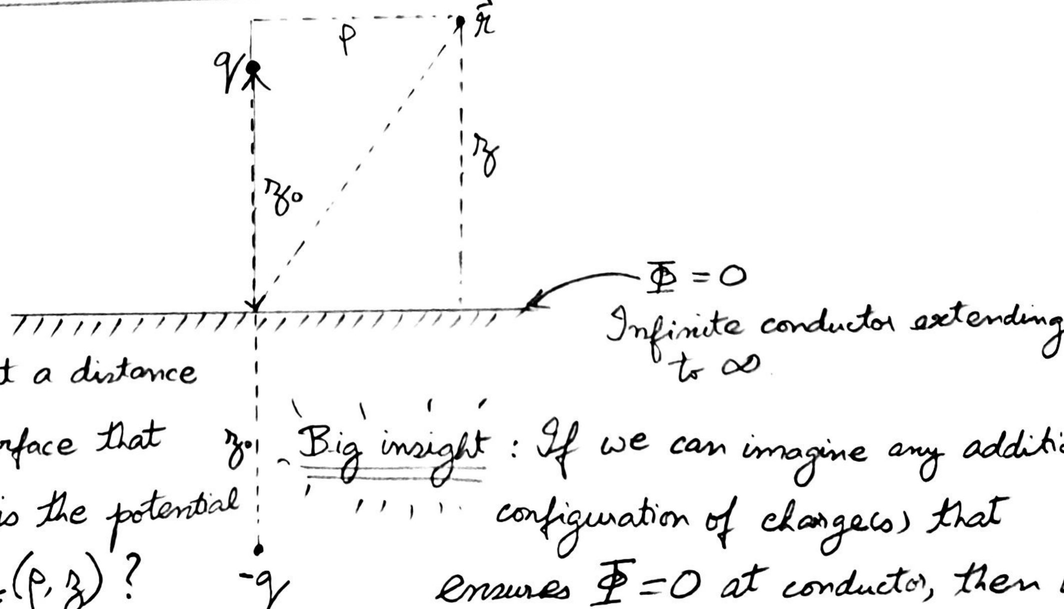
$\Phi_0(\vec{r})$ → Solution to the Laplace Eqn, $\nabla^2 \Phi = 0$

General solution of $\nabla^2 \Phi = \frac{q}{\epsilon_0}\delta^3(\vec{r}-\vec{r}_0)$

The full solution $\Phi(\vec{r})$ has to match boundary conditions when, say, a conductor of potential $\Phi_0(\vec{r}')$ is present in the problem.

Uniqueness theorem proven in Lecture 10 tells us we only need to find ONE solution that satisfies boundary conditions.

Simple Problem: Method of Images Example 1



We have a charge q at a distance z_0 from a conducting surface that extends to ∞. What is the potential Φ at an arbitrary point $\vec{r} = (\rho, z)$?

Big insight: If we can imagine any additional configuration of charge(s) that ensures $\Phi = 0$ at conductor, then we are set! By Uniqueness!

So, we imagine a charge $-q$ at $-z_0$, i.e on "other side" of the conductor.
It is obvious that this makes $\Phi = 0$ at the conductor surface.

Thus, we can write the $\Phi(\vec{r} = \rho, z)$ as:

$$\Phi(\rho, z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{\rho^2 + (z - z_0)^2}} - \frac{1}{\sqrt{\rho^2 + (z + z_0)^2}}\right]$$

From this, we can derive the $\vec{E} = -\vec{\nabla}\phi$ [in cylindrical coordinates]

$$\vec{E}(\rho, z) = \frac{q}{4\pi\epsilon_0}\left[\frac{\rho\hat{\rho} + (z - z_0)\hat{z}}{(\rho^2 + (z - z_0)^2)^{3/2}} - \frac{\rho\hat{\rho} + (z + z_0)\hat{z}}{(\rho^2 + (z + z_0)^2)^{3/2}}\right]$$

This could be simplified. But hey, can we calculate the induced charge density on the surface of the conductor?
Yes! We can work that out from the $\vec{E}$ field at the surface because $\boxed{\vec{E} = \frac{\sigma}{\epsilon_0}}$.

So, the $\vec{E}(\rho, 0) = \frac{q}{4\pi\epsilon_0}\left[\frac{2z_0\hat{z}}{(\rho^2 + z_0^2)^{3/2}}\right]$

$$\vec{E}(\rho, 0) = \frac{q z_0 \hat{z}}{2\pi\epsilon_0 (\rho^2 + z_0^2)^{3/2}}$$

$$\therefore \boxed{\sigma(\rho, 0) = \epsilon_0 E = \frac{q z_0}{2\pi(\rho^2 + z_0^2)^{3/2}}}$$

All this charge was drawn up from the ground since the conductor must have had to be grounded in order to get $\Phi = 0$ on it. So, what is the total charge that got drawn up from the ground? Integrate $\sigma(\rho)$ over the plane.

$$q' = \int_0^\infty \sigma(\rho) \cdot 2\pi\rho \, d\rho = \frac{q z_0}{2\pi}\int_0^\infty \frac{2\pi\rho}{(\rho^2 + z^2)^{3/2}} d\rho$$

$t = \rho^2 + z^2$
$dt = 2\rho \, d\rho$
Substitute t for ρ ...

$$q' = -q$$

So, equal and opposite charge was drawn up!

What would the attractive force on the charge q be due to conductor?

Well, integrate the force due to rings of induced charge on the plane from $\rho = 0$ to ∞.

$$\vec{F}(z_0) = \frac{\hat{z}\,q}{2\epsilon_0} \int_{\rho=0}^{\infty} 2\pi\rho\, d\rho\, \sigma(\rho) \cdot \frac{z_0}{(z_0^2 + \rho^2)^{3/2}}$$

$$\vec{F} = -\hat{z} \frac{q^2}{4\pi\epsilon_0 (2z_0)^2}$$ ← it is as if it is getting pulled by the image charge at a distance $2z_0$ away!

So, is everything exactly the same as if there were an image charge under the $z=0$ plane? Almost... but not quite the energy!

Naively, you'd think the potential energy of the system would be:

$$U = \frac{q^2}{4\pi\epsilon_0 (2z_0)^2}$$

But it would be half of that actually because... and there are 2 ways to think about it:

1. No work needs to be done to move the imaginary charge because the imaginary charge is standing in for all the induced charge... and since the induced charges were moving on a $\Phi = 0$ surface at every moment, they did no work. Or rather, no work was done on them to get them to move into their final configuration.
2. The energy is stored in the fields with density $\sim \frac{1}{2}\epsilon_0 E^2$ and unlike the situation of a real charge, there is just "half the space" above $z=0$ where the $\vec{E}$ field actually exists.

Real life example

Consider an electron very near to a metal surface. It is going to get attracted to the surface. But, quantum mechanics will prevent it from being exactly on the surface because then we would know its z-position with infinite precision! So, it must sort-of hover over the surface of the metal like in energy bands. Really?!! Can you work out the quantum mechanical structure of an electron next to a metal, or infinite $\Phi = 0$, surface?

Well, the Schrödinger equation only cares about the potential energy.

$$i\hbar \frac{\partial \psi}{\partial t} = \left[\frac{\hat{p}^2}{2m} + V\right] \psi$$

If we just care about energy levels, we can just consider the time-independent equation to get energy eigenstates.

$$\hat{H}\psi = E\psi$$

$$\therefore \left[\frac{p^2}{2m_e} + V\right]\psi = E\psi$$

And we just found out, $V = \frac{q^2}{4\pi\epsilon_0 (2z)^2} \times \frac{1}{2}$

So,

$$\frac{-\hbar^2}{2m_e}\psi(z) + \frac{q^2 \psi}{32\pi\epsilon_0 z^2} = E\psi$$

This is exactly the same equation as the radial structure of a hydrogen atom. You do get energy bands with energy

$$\text{Energy}(n) = -\left(\frac{m_e e^4}{2\hbar^2}\right)\frac{1}{n^2}$$

You get something like a Balmer spectrum!