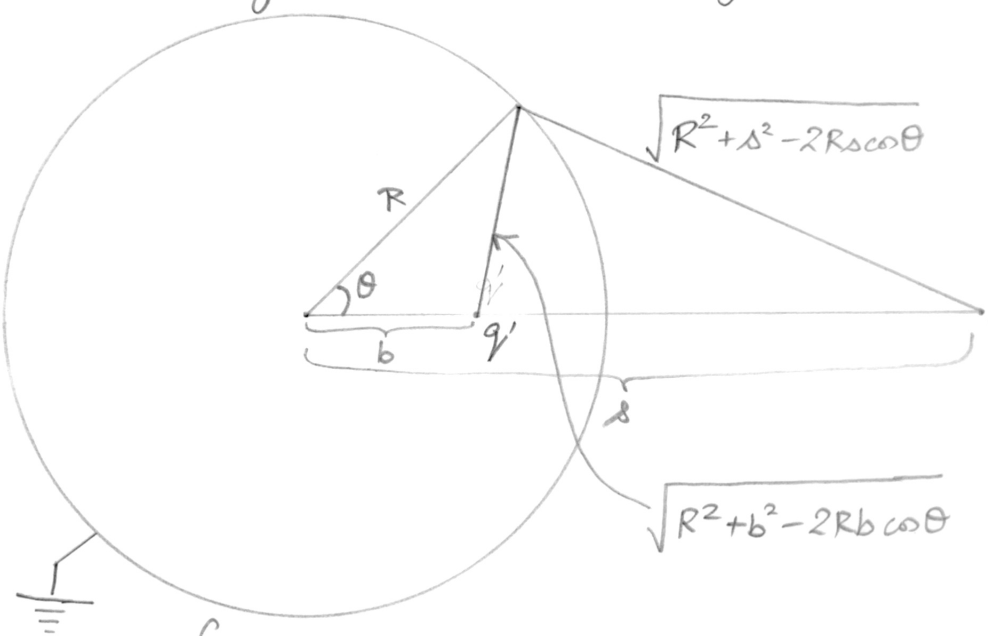
Lecture 13 Solving Poisson's Equation using Method of Images



What are q' and b?

Grounded conducting sphere A charge "q" is placed near a grounded metal sphere

We are asked to find the potential at any arbitrary point near the sphere.
Physically what would happen is the sphere will draw up opposite charge from the ground. But not necessarily equal. Just enough to ensure the surface of the conductor is at 0 potential, i.e $\Phi = 0$.

One could try to solve Poisson's Equation with source q and the $\Phi = 0$ boundary condition on the surface of the sphere.

Or, one could get creative and imagine a mirror charge q' at b from center of the sphere "sitting in" for the charge drawn up from the ground.
Then, due to this imaginary mirror charge, the Φ at any point on the shell is

$$\Phi(\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{R^2+s^2-2Rs\cos\theta}} + \frac{q'}{\sqrt{R^2+b^2-2Rb\cos\theta}}\right] = 0$$

So,

$$\frac{q^2}{R^2+s^2-2Rs\cos\theta} = \frac{q'^2}{R^2+b^2-2Rb\cos\theta}$$

$$q^2(R^2+b^2-2Rb\cos\theta) = q'^2(R^2+s^2-2Rs\cos\theta)$$

Thanks to Scott Demarest

Since this must be true for all θ, we can equate the θ-dependent parts and separately equate the θ-independent parts

Equating the θ-dependent parts,

$$2q^2 R b \cos\theta = 2q'^2 R s \cos\theta$$

$$\therefore \quad q' = q\sqrt{\frac{b}{s}} \qquad \text{——} \quad 1$$

Proof due to Scott Demarest

Equating the θ-independent parts,

$$q^2(R^2+b^2) = q'^2(R^2+s^2)$$

Using 1,
$$\cancel{q^2}(R^2+b^2) = \cancel{q^2}\frac{b}{s}(R^2+s^2)$$

$$\therefore \quad b^2 - \left(\frac{R^2}{s}+s\right)b + R^2 = 0$$

Solving this quadratic equation,

$$b = \frac{\left(\frac{R^2}{s}+s\right) \pm \sqrt{\left(\frac{R^4}{s^2}+s^2+2R^2\right)-4R^2}}{2}$$

$$= \frac{\left(\frac{R^2}{s}+s\right) \pm \sqrt{\frac{R^4}{s^2}-2R^2+s^2}}{2}$$

$$b = \frac{\left(\frac{R^2}{s}+s\right) \pm \sqrt{\left(\frac{R^2}{s}-s\right)^2}}{2}$$

So,

$$b \to \left(\frac{R^2}{s}+\cancel{s}+\frac{R^2}{s}-\cancel{s}\right)/2 = \frac{R^2}{s} \quad \leftarrow \text{Physical solution}$$

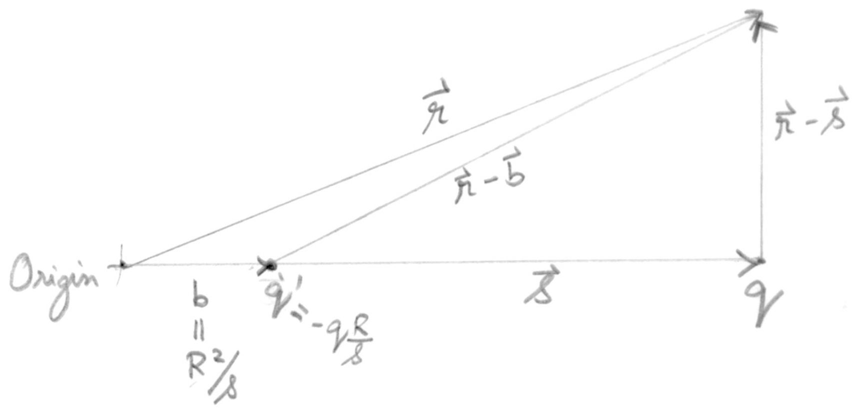
$$b \to \left(\cancel{\frac{R^2}{s}}+s-\cancel{\frac{R^2}{s}}+s\right)/2 = s \quad \leftarrow \text{Unphysical solution.}$$

This is saying paste the −ve charge on top of the +ve charge, equal and opposite. Thus $\Phi = 0$ everywhere!

So, $\boxed{b = \frac{R^2}{s}}$

Substituting this back in 1, $\boxed{q' = -q\frac{R}{s}}$ ← Only opposite charge makes sense!

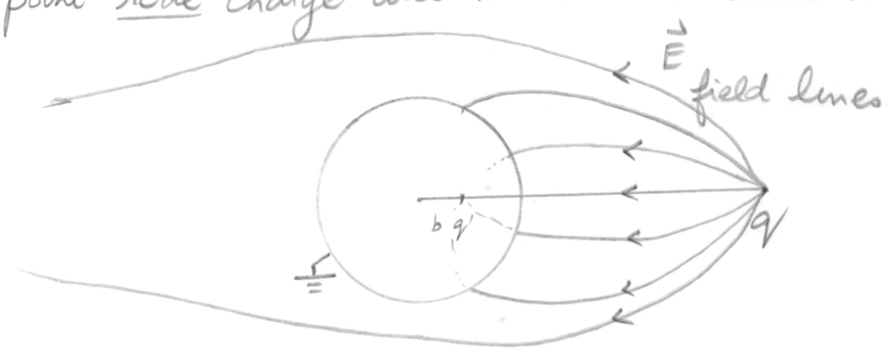
So, now we can write the potential at any arbitrary point outside the sphere by pretending there are only the real charge and imaginary charge.



$$\boxed{\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{r}-\vec{s}|} - \frac{(R/s)}{\left|\vec{r}-\vec{s}\,\frac{R^2}{s^2}\right|}\right]}$$

The Solution!

Because q' is not $-q$, but less, field lines starting from the point real charge will not all terminate on the sphere!



Force between point charge and conductor will be attractive.

$$F = \frac{qq'}{4\pi\epsilon_0(s-b)^2} = \frac{q^2(R/s)}{4\pi\epsilon_0\left(s-\frac{R^2}{s}\right)^2} = \frac{q^2 Rs}{4\pi\epsilon_0(s^2-R^2)^2}$$

This falls off as $1/s^3$ Dipolar!