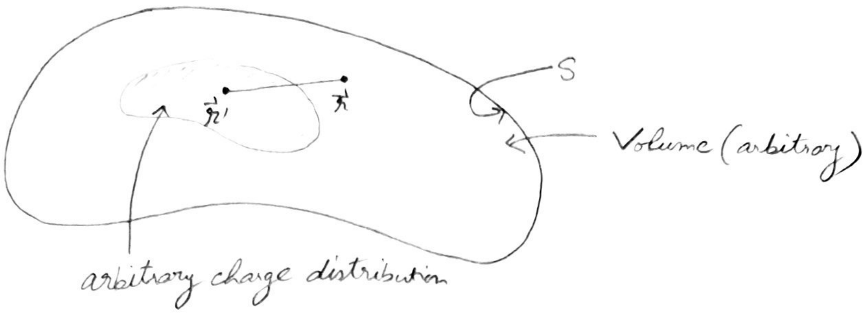
Lecture 14

Poisson's Equation — Solving it with the Green's Function.

$$\nabla^2 \Phi(\vec{r}) = \frac{-\rho(\vec{r})}{\epsilon_0} \quad \text{in volume } V$$



Φ or $\frac{\partial \Phi}{\partial n}$ is specified on S.

We exploit a function $G(\vec{r}, \vec{r}')$ that satisfies Poisson's Eqn for a deltafunction source.

$$\nabla^2 G(\vec{r}, \vec{r}') = \frac{-\delta^3(\vec{r} - \vec{r}')}{\epsilon_0} \quad ; \quad \vec{r}, \vec{r}' \in V$$

An <u>aside</u> :

$$\int \vec{\nabla} \cdot \vec{F} \, dV = \oint \vec{F} \cdot d\vec{S} \qquad \text{(Gauss' Law)}$$

If $\vec{F} = \phi(\vec{r}) \vec{\nabla} \psi(\vec{r})$

Then, $\vec{\nabla} \cdot \vec{F} = (\vec{\nabla}\phi) \cdot (\vec{\nabla}\psi) + \phi \nabla^2 \psi$

So, $$\int (\vec{\nabla}\phi) \cdot (\vec{\nabla}\psi) \, dV + \int \phi \nabla^2 \psi \, dV = \oint \phi \vec{\nabla}\psi \cdot d\vec{S} \quad \leftarrow \text{Green's first identity}$$

Swapping ϕ & ψ : $$\ominus \int (\vec{\nabla}\psi) \cdot (\vec{\nabla}\phi) \, dV + \ominus \int \psi \nabla^2 \phi \, dV = \ominus \oint \psi \vec{\nabla}\phi \cdot d\vec{S}$$

Subtracting : $$\int [\phi \nabla^2 \psi - \psi \nabla^2 \phi] \, dV = \oint [\phi \vec{\nabla}\psi - \psi \vec{\nabla}\phi] \cdot d\vec{S} \quad \leftarrow \text{Green's second identity.}$$

To apply this, Let $\phi = \Phi(\vec{r}') \longleftarrow$ potential from volume charge distribution $\rho(\vec{r}')$

And $\psi = G(\vec{r}', \vec{r}) = \frac{1}{|\vec{r} - \vec{r}'|}$

Then, $$\int \Phi(\vec{r}') \nabla^2 G(\vec{r}', \vec{r}) \, dV' - \int G(\vec{r}', \vec{r}) \nabla^2 \Phi(\vec{r}') \, dV' = \oint \Phi(\vec{r}') \vec{\nabla} G \cdot d\vec{S}' - \oint G(\vec{r}', \vec{r}) \vec{\nabla}\Phi \cdot d\vec{S}'$$

Now, $\nabla^2 G(\vec{r}, \vec{r}') = \dfrac{-\delta^3(\vec{r} - \vec{r}')}{\epsilon_0}$

$$-\frac{\Phi(\vec{r})}{\epsilon_0} - \int_V G(\vec{r}', \vec{r}) \frac{\rho(\vec{r}')}{\epsilon_0} dV' = \oint \Phi(\vec{r}') \frac{\partial G}{\partial \eta} \cdot dS' - \int G(\vec{r}', \vec{r}) \frac{\partial \Phi}{\partial \eta} dS'$$

$$\boxed{\Phi(\vec{r}) = \int_V \rho(\vec{r}')\, G(\vec{r}', \vec{r})\, dV' - \epsilon_0 \int_S \Phi(\vec{r}') \frac{\partial G}{\partial \eta} dS' + \epsilon_0 \int G(\vec{r}', \vec{r}) \frac{\partial \Phi}{\partial \eta} dS}$$

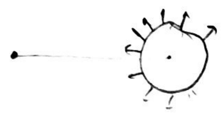
Coulomb potential for $\vec{G}(\vec{r}, \vec{r}') = \dfrac{1}{|\vec{r} - \vec{r}'|}$

for $\vec{G}(\vec{r}, \vec{r}') = \dfrac{1}{|\vec{r} - \vec{r}'|}$

$\dfrac{\partial G}{\partial \eta'} \sim \dfrac{+\hat{n}\cdot(\vec{r}-\vec{r}')}{r^2}$ ... dipole layer of strength $\Phi(\vec{r}')$

this term only contributes if point is within a conducting surface. Basically the background potential inside the surface S. Φ_0

$\dfrac{\partial \Phi}{\partial \eta} = E_\perp$ to the surface. In case the surface gets charged up.



Green's function has the most general form

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} + \underline{F(\vec{r}, \vec{r}')}$$

→ potential from rearrangement of charges.

where $F(\vec{r}, \vec{r})$ satisfies $\nabla'^2 F(\vec{r}, \vec{r}')$ throughout the volume.

↳ has enough "give" to satisfy boundary conditions Dirichlet or Neumann.

If we can find a $G(\vec{r}, \vec{r}')$ that is 0 when $\vec{r}'$ is on S, and we know its normal derivative $\dfrac{\partial G}{\partial \eta'}$ at $\vec{r}' = S$, then we can find $\Phi(\vec{r})$ from $\rho(\vec{r}')$ and $\Phi(\vec{r}')$.