Lecture 15 — Green's Function solution – An Example.

We are trying to solve Poisson's Equation with non-trivial boundary conditions.

$$\nabla^2 \Phi(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}$$

We use the Green's function defined as:

$$\nabla^2 G(\vec{r},\vec{r}\,') = -\frac{\delta^3(\vec{r}-\vec{r}\,')}{\epsilon_0} \quad ; \quad \vec{r}, \vec{r}\,' \in V$$

We need to solve for the Green's function for a given geometric situation and use linear combinations of it to find the solution for $\rho(\vec{r})$.

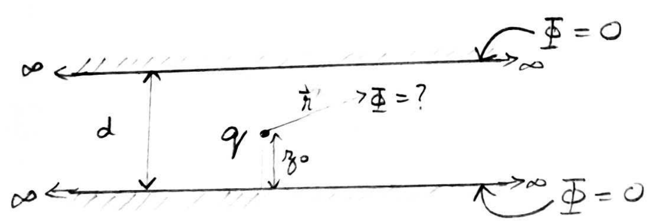
From the Green's function, we saw in Lecture 14:

$$\Phi(\vec{r}) = \int_V \rho(\vec{r}\,') G(\vec{r},\vec{r}\,')\, dV' - \epsilon_0 \int_S \Phi(\vec{r}\,') \frac{\partial G}{\partial n}\, dS' + \epsilon_0 \int_S G(\vec{r}\,',\vec{r}) \frac{\partial \Phi}{\partial n}\, dS'$$

also called a "propagator" in field theory because it propagates cause at $\vec{r}\,'$ to effect at $\vec{r}$.

If we just know Φ at S', we need this $G(\vec{r},\vec{r}\,')$ at $S'=0$ to "cover up" our ignorance of $\frac{\partial \Phi}{\partial n}$. ☺
This is key.

Example



Consider a charge between two metal conductors separated by distance d. Conductors are grounded, $\Phi = 0$. Charge is z_0 from one of the plates. What is the Φ due to the charge at any arbitrary point inside?

What symmetry / coordinate system should we use?

- Cartesian?
- Cylindrical? ⟵ THIS ONE!
- Spherical?

Separation of variables in Cylindrical coordinates for the Laplace Equation.

We need this aside!

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

Assume, $\Phi(\rho, \phi, z) = R(\rho)X(\phi)Z(z)$

Then,

$$XZ\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{RZ}{\rho^2}\frac{d^2X}{d\phi^2} + RX\frac{d^2Z}{dz^2} = 0$$

$\Rightarrow$ Multiplying by $\frac{\rho^2}{\Phi}$,

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \underbrace{\frac{1}{X}\frac{d^2X}{d\phi^2}}_{\substack{-m^2 \\ \text{oscillatory}}} + \underbrace{\frac{\rho^2}{Z}\frac{d^2Z}{dz^2}}_{\substack{\to -k^2 \text{ for oscillatory} \\ \to +k^2 \text{ for exponential}}} = 0$$

$-k^2$ for oscillatory, $+k^2$ for exponential } Depending on boundary conditions

So, $$\frac{d^2X}{d\phi^2} = -m^2X$$; m takes integer values

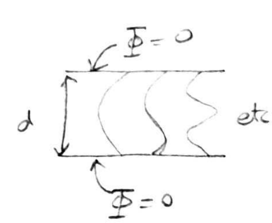
$$\Rightarrow X(\phi) = A\cos(m\phi) + B\sin(m\phi)$$

Oscillatory solution in ϕ

In $Z(z)$, we can opt for oscillatory of exponential solutions.

- Oscillatory solutions are appropriate for periodic boundary conditions or when we have $\Phi = 0$ b.c. like an infinite well.

In our target problem, this is what we need.



So, $$\frac{d^2Z}{dz^2} = -k^2Z$$

$$\Rightarrow Z(z) = C\cos(kz) + D\sin(kz)$$

For the radial part:

If we chose $Z(z)$ to be oscillatory

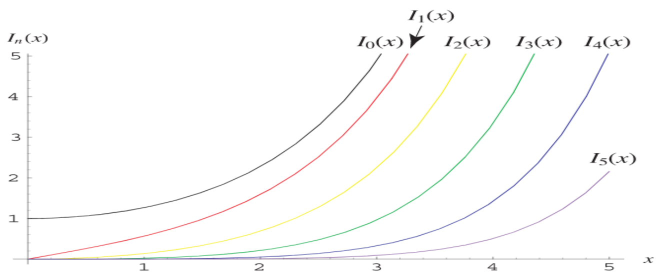
$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - m^2 - \rho^2 k^2 = 0$$

$$\therefore \frac{\rho}{R}\frac{dR}{d\rho} + \frac{\rho^2}{R}\frac{d^2R}{d\rho^2} - (m^2 + \rho^2 k^2) = 0$$

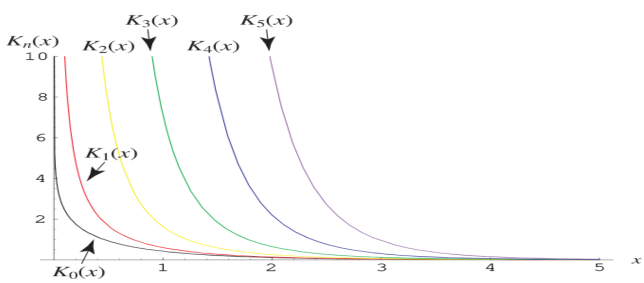
$$\therefore \rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} - (\rho^2 k^2 + m^2)R = 0$$ ← Modified Bessel Differential Eqn

$$\Rightarrow R(\rho) = E\, I_m(\rho k) + F\, K_m(\rho k)$$

$I_m(\rho k)$ = Modified Bessel function of first kind.



$K_m(\rho k)$ = Modified Bessel function of second kind.



You generally have to use both, one inside the cylindrical conductor, and the other outside it. Kind of like

$$Ar^n + \frac{B}{r^{n+1}}$$

inside   outside.

Tiny Scanner

⊙ For Z, we can also choose exponential solutions. These are appropriate for open boundary conditions.

So, $$\frac{d^2 Z}{dz^2} = k^2 Z$$

$$\Rightarrow Z(z) = C e^{kz} + D e^{-kz}$$

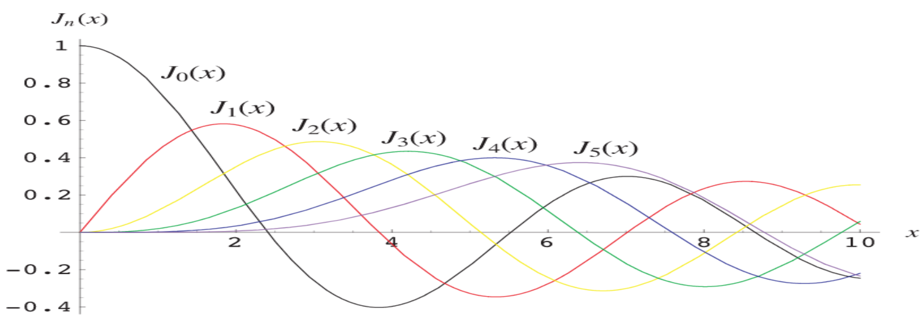
But this choice has an effect on the radial solutions.

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) - m^2 + k^2\rho^2 = 0$$

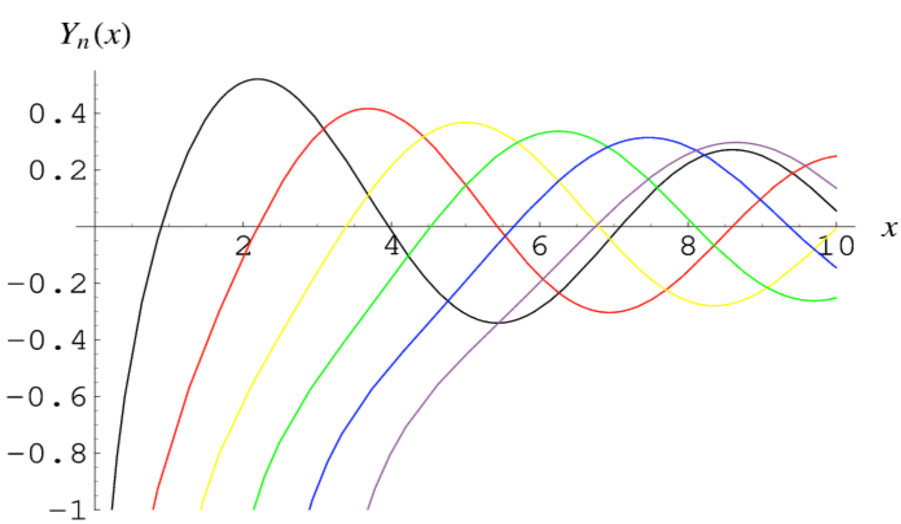
$$\therefore \rho^2 \frac{d^2R}{d\rho^2} + \rho \frac{dR}{d\rho} + (\rho^2 k^2 - m^2) R = 0 \longleftarrow \text{Bessel Differential Equation.}$$

$$\Rightarrow R(\rho) = E\, J_m(\rho k) + F\, Y_m(\rho k)$$

$J_m(\rho k)$ = Bessel function of the first kind.



$Y_m(\rho k)$ = Bessel function of the second kind.



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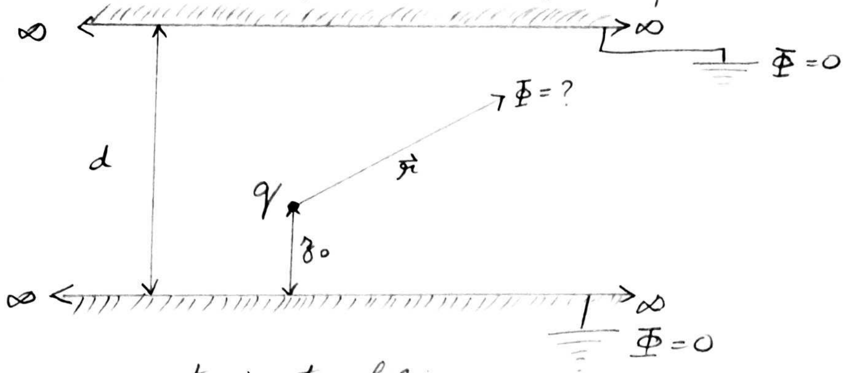
Now we can try to solve for the $G(\vec{r}, \vec{r}\,')$ relevant for our problem.



Reminder, we are trying to solve

$$\nabla^2 G(\vec{r}, \vec{r}\,') = \frac{-\delta^3(\vec{r} - \vec{r}\,')}{\epsilon_0}$$

That is, in cylindrical coordinates,

$$\nabla^2 G(\rho, \phi, z; \rho', \phi', z') = -\frac{\delta}{\epsilon_0}(\rho - \rho')\,\delta(\phi - \phi')\,\delta(z - z')$$

We can express these delta-functions int terms of the typical solutions and exploit their orthoganalities!

$$\text{So,}\quad \delta(\phi - \phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} \begin{cases} \to \text{at } \phi = \phi' \Rightarrow \infty \\ \to \text{at } \phi \neq \phi' \Rightarrow 0 \end{cases}$$

For the $Z(z)$ we should choose oscillatory, $\sin(kz)$ solutions.

$$\text{So,}\quad \delta(z - z') = \frac{2}{d}\sum_{k=1}^{\infty} \sin\left(\frac{k\pi z}{d}\right)\sin\left(\frac{k\pi z'}{d}\right)$$

And since we chose oscillatory solutions for Z, we must choose modified Bessel functions for R.

$$\delta(\rho - \rho') = \sum_{m=-\infty}^{\infty}\big(I_m(\rho k) + K_m(\rho k)\big)\big(I_m(\rho' k) + K_m(\rho' k)\big)$$

So, our Greens function is of this form. (assumption)

$$G(\vec{r}, \vec{r}\,') = \frac{1}{\epsilon_0 d \pi}\sum_{k=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} \sin\left(\frac{k\pi z}{d}\right)\sin\left(\frac{k\pi z'}{d}\right)\left[I_m(\rho k) + K_m(\rho k)\right]\left[I_m(\rho' k) + K_m(\rho' k)\right]$$

Substituting this back into $\nabla^2 G(\vec{r},\vec{r}') = -\frac{\delta^3(\vec{r}-\vec{r}')}{\epsilon_0}$

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

So,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) \equiv \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\rho^2}$$

$$\frac{\partial^2 G}{\partial\rho^2} \Rightarrow \frac{\partial^2(I_m + K_m)}{\partial\rho^2}\cdot\underbrace{\left[\quad\right]}_{\text{rest of } G}\cdot k^2$$

$$\frac{1}{\rho}\frac{\partial G}{\partial\rho} \Rightarrow \frac{1}{\rho}\frac{\partial(I_m + K_m)}{\partial\rho}\left[\quad\right]k$$

$$\frac{1}{\rho^2}\frac{\partial^2 G}{\phi^2} \Rightarrow \frac{-m^2}{\rho^2}G$$

$$\frac{\partial^2}{\partial z^2} \Rightarrow -k^2 G$$

Plugging it back in: $\frac{1}{\rho}\frac{\partial(I_m+K_m)}{\partial\rho}\left[\quad\right]k + \frac{\partial^2(I_m+K_m)}{\partial\rho^2}\left[\quad\right]k^2 - \frac{m^2 G}{\rho^2} - k^2 G$

$$= 0 \text{ if } \vec{r} \neq \vec{r}'$$
$$= \infty \text{ if } \vec{r} = \vec{r}'$$

To find exact form of $G(\vec{r},\vec{r}')$, we must use intelligence!

Put the charge at $(x_0, y_0, z_0) = (0, 0, z_0)$, evaluate it there. $\underline{m=0}$ (azimuthally symmetric)

$$G(\vec{r}) = \frac{1}{\epsilon_0 \pi d}\sum_{k=1}^{\infty}\sin\left(\frac{k\pi z}{d}\right)\sin\left(\frac{k\pi z_0}{d}\right)\left[I_m(0) + K_m(0)\right]\left[I_m\left(\frac{\rho k\pi}{d}\right)\cdot K_m\left(\frac{\rho k}{d}\right)\right]$$

For $m=0$, $I_0(0) = \infty$; Therefore only K_0 exist.

$$\therefore G(\vec{r}) = \frac{1}{\epsilon_0 \pi d}\sum_{k=1}^{\infty}\sin\left(\frac{k\pi z}{d}\right)\sin\left(\frac{k\pi z_0}{d}\right)K_0\left(\frac{k\pi\rho}{d}\right)$$

Given this $G(\vec{r}, \vec{r}')$, we can get the form of $\Phi(\vec{r})$ immediately:

$$\boxed{\phi(\rho, z) = \frac{q}{\epsilon_0 \pi d} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{d}\right) \sin\left(\frac{n\pi z_0}{d}\right) K_0\left(\frac{n\pi\rho}{d}\right)}$$

What is the charge induced on the $z = d$ plate?

$$\varphi(d) = \epsilon_0 \cdot 2\pi \cdot \int_0^{\infty} \rho \cdot \underbrace{\left.\frac{\partial \Phi}{\partial z}\right|_{z=d}}_{\text{Electric field.}} d\rho$$

$$\boxed{\varphi = \frac{-z_0 q}{d}}$$