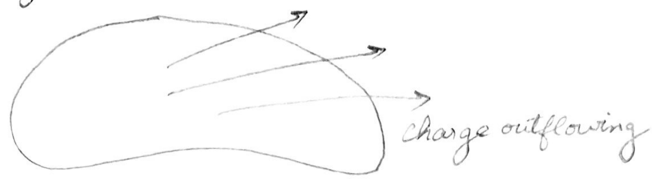
Lecture 16 Steady Current

Equation of continuity



Charge flowing out of the volume $= \oint_S \vec{j} \cdot d\vec{S}$ → current density $\boxed{\vec{j} = \frac{dI}{dA}}$

This should equal the decrease in charge: $-\frac{dq}{dt}$ → $q = \int \rho \, dV$

So, $$\frac{\partial}{\partial t} \int_V \rho \, dV = -\oint_A \vec{j} \cdot d\vec{S}$$

$$\therefore \int_V \frac{\partial \rho}{\partial t} dV = -\int_V (\vec{\nabla} \cdot \vec{j}) \, dV$$

$$\therefore \boxed{\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{j}}$$

Equation of continuity. Expresses the law of conservation of charge. Equally applicable for incompressible fluids, heat, energy, anything with a non-zero chemical potential.

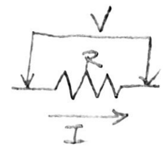
Steady currents refer to divergence-free current densities. That is, charges are not accumulating or depleting from any point.

$\boxed{\vec{\nabla} \cdot \vec{j} = 0}$ — Condition for steady current

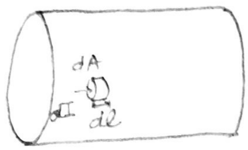
Current in Neutral Matter

Ohm's Law

Established empirically that $V = IR$



What would be the differential, general form for this?



For the differential element,

$$dI = \frac{dV}{R}$$

But $R = \rho \frac{dl}{dA} \Leftarrow$ Comes from $R = \frac{\rho l}{A}$ — resistivity

So, $$dI = \frac{dV}{\rho \frac{dl}{dA}}$$

$$\therefore \frac{dI}{dA} = \frac{1}{\rho}\frac{dV}{dl}$$

$$\therefore \boxed{\vec{j} = \frac{1}{\rho}\vec{E}} \equiv \boxed{\vec{j} = \sigma \vec{E}}$$

conductivity $= 1/\rho$

true at every point in the conducting matter.

Microscopic origin of Ohm's Law

To begin to understand that, we need to first recognize

$$\boxed{\vec{j}(\vec{r}) = \rho(\vec{r})\,\vec{v}(\vec{r})}$$

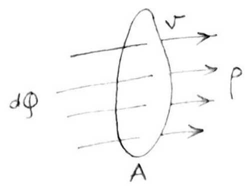
current density

charge density

charge velocity

Why is this true?

Consider any surface through which charge is passing.



In time dt, the amount of charge that passes through is:

$$dq = A v dt \rho$$

$$\therefore \frac{1}{A}\frac{dq}{dt} = j = v\rho$$

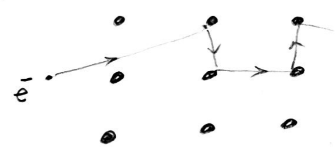
$$\therefore \vec{j}(\vec{r}) = \vec{v}(\vec{r})\rho(\vec{r})$$

What is the velocity of charged particles in neutral matter?

Drift velocity

"Drude Model"

Classical, billiard-ball model.



The electron is going to bump into a nucleus every τ time.

When it does, it loses initial direction, and on average its new velocity is 0.

The velocity it achieves in time τ is $a\tau$. ← "mean free time"

$$a = \frac{F}{m_e} \leftarrow \text{mass of the electron.}$$

$$F = qE$$

q ← charge of the electron.

So, $\boxed{v_{drift} = \frac{qE}{m_e}\tau}$ drift velocity $\propto$ Electric force.

often written as $v_d = \mu q E$, $\mu \to$ mobility $= \frac{\tau}{m_e}$

Explanation of Ohm's Law

$$j = \frac{qE}{m_e}.\tau.nq$$

n ← number density of charges.

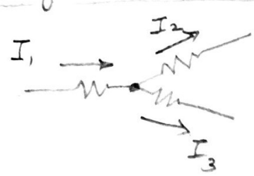
$$\therefore \boxed{j = n\frac{q^2\tau}{m_e}E} \leftarrow \text{Ohm's law.}$$

$$j = \frac{nq^2\tau}{m_e} E$$

This suggests, $\sigma = \frac{1}{\rho} = \frac{nq^2\tau}{m_e}$ ← microscopic origin of resistivity. Does it make intuitive sense?

Kirchhoff's Laws for Ohmic circuits

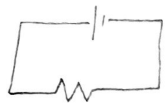
1.



$I_1 + I_2 + I_3 = 0$

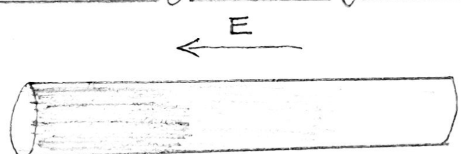
$$\sum_k I_k = 0$$

2.



$\sum_k V_k = \sum_n I_n R_n$ in a closed loop.

Einstein Relation between charge diffusivity and charge mobility



Charges diffusing in 1D
from high charge density on left to low density on right
kept in equilibrium by an external electric field $\vec{E}$ to counter diffusion.

⊙ What is the current density due to diffusion?

$$\frac{\partial \rho}{\partial t} = D\nabla^2\rho$$ ← Diffusion equation for charge density ρ.

Since, $\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{j}$ ← Continuity equation

$$\therefore \quad \vec{\nabla}\cdot\vec{j} = -D\,\vec{\nabla}\cdot(\vec{\nabla}\rho)$$

$$\therefore \boxed{\vec{j} = -D\vec{\nabla}\rho}$$ ← Fick's Law

⊙ What is the current density due to $\vec{E}$?

$$\vec{j} = -nq^2\frac{\tau}{m_e}E$$

$$= -(nq)(qE)\left(\frac{\tau}{m_e}\right)$$

$$\boxed{\vec{j} = -\rho F\mu}$$

Set these to be equal and opposite

$$-D\vec{\nabla}\rho - \rho\mu\vec{F} = 0$$

Now, $\vec{F} = q\vec{E} = q\vec{\nabla}\Phi$ ← external voltage applied

$$\therefore -D\vec{\nabla}\rho - \rho\mu q\vec{\nabla}\Phi = 0$$

$$D\vec{\nabla}\rho + \rho\mu q\frac{d\Phi}{d\rho}\vec{\nabla}\rho = 0$$

$$\therefore D + \rho\mu q\frac{d\Phi}{d\rho} = 0$$

$$\therefore D = -\rho\mu q\frac{d\Phi}{d\rho}$$

Since this is in thermodynamic equilibrium,

$$\rho(\vec{x}) = A e^{-q\Phi(\vec{x})/kT}$$

$$\therefore \frac{d\rho}{d\Phi} = -\frac{Aq}{kT}e^{-q\Phi/kT} = \frac{-\rho q}{kT}$$

$$\therefore D = +\rho\mu q\frac{kT}{\rho q}$$

$$\therefore \boxed{D = \mu kT}$$ ← Einstein discovered this in 1905 in the context of Brownian motion.

→ D relates to the statistical fluctuations in the medium.

→ μ relates to the mobility or 1/drag in the medium.

These are intimately connected!

- Holds true ACROSS PHYSICS (almost... violated in glasses)
- Precursor of the Fluctuation-Dissipation Theorem.
  - ↳ Thermal fluctations of any observable is directly related to its susceptibility.
- Holds true in Classical and Quantum statistical mechanics