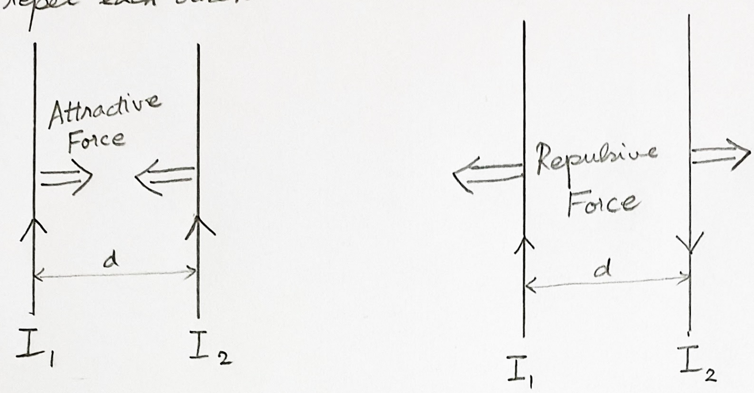
Lecture 17 Magnetostatics Basics

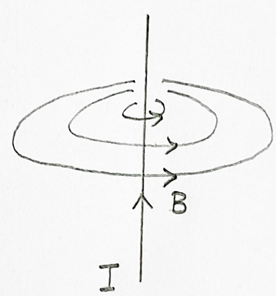
⊙ How do we know moving charges influence each other due to their motion, i.e. above and beyond electrostatic interactions?

→ Many ways, one obvious way is that neutral, current carrying conductors attract or repel each other.



$$F \propto \frac{I_1 I_2}{d}$$

Magnetic field due to a wire:



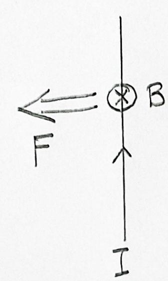
Force on a wire due to a magnetic field:

"Fleming's Left Hand rule"

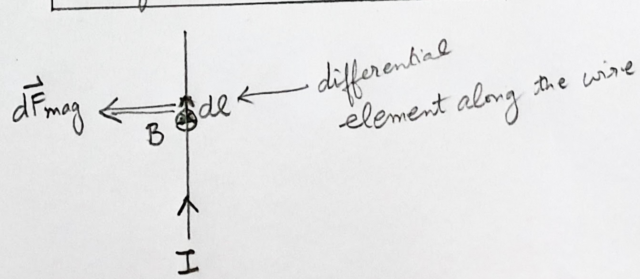
Index finger = B field

Midde finger = Current

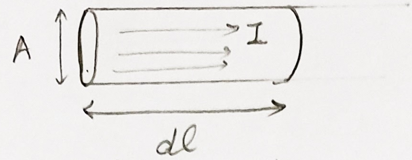
implies Thumb = Force.



Lorentz force law: $\boxed{d\vec{F}_{magnetic} = I(d\vec{l} \times \vec{B})}$



Can we extend this to moving charges?



$$I\,d\vec{l} = jA\,d\vec{l} \quad \left(\text{because } j = \frac{I}{A}\right)$$
$$= \rho v A\,d\vec{l} \quad (\text{because } j = \rho v)$$
$$= \vec{v}\rho\,dV \quad (\text{because } A\,dl = dV$$
and direction is that of charge velocity)

$$\therefore \quad I\,d\vec{l} = \vec{v}\,dq \quad (\text{because } \rho\,dV = dq)$$

So, $$d\vec{F}_{mag} = I(d\vec{l} \times \vec{B}) = dq(\vec{v} \times \vec{B})$$

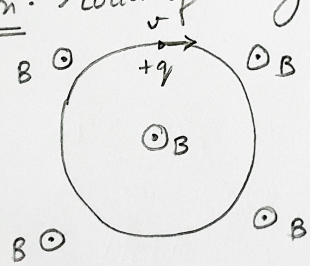
In other words, $$\boxed{\vec{F}_{mag} = q(\vec{v} \times \vec{B})}$$

Also the Lorentz force law

More generally, under the influence of an $\vec{E}$ and $\vec{B}$ field:

$$\boxed{\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})}$$ ← Also called the Lorentz force law.

Cyclotron Motion: Motion of a charged particle under its own inertia.



> Magnetic field out of plane, B

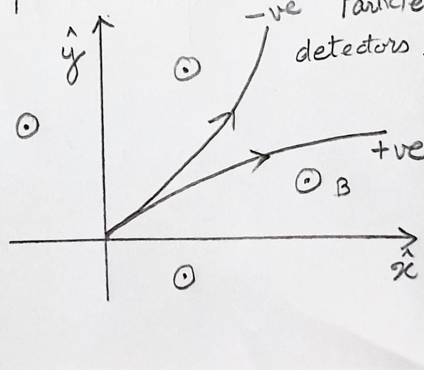
> Particle with charge q, mass m.

Centripetal acceleration due to B:

$$F = qvB$$

This results in circular path with radius R:

$$\frac{mv^2}{R} = F$$



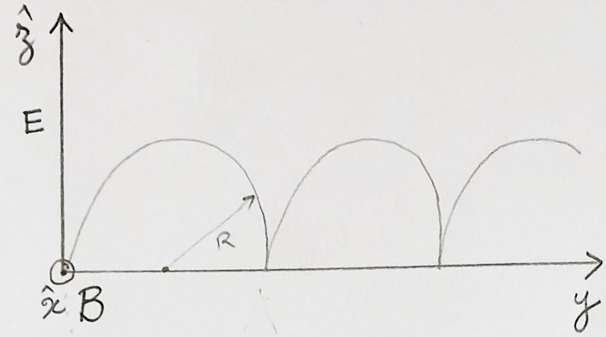
Particle physics detectors exploit this.

$$\therefore \quad \frac{mv^2}{R} = qvB$$

$$\therefore \quad \boxed{p = qBR}$$

momentum of particle ⟸ radius of curvature

Cycloid motion in a crossed $\vec{E}$ & $\vec{B}$ field.



- Electric field $\vec{E}$ in $\hat{z}$ direction
- Test charge q at $(0,0,0)$
- Magnetic field $\vec{B}$ in $\hat{x}$ direction.

What happens? Particle moves up along $\hat{z}$ due to $\vec{E}$ field initially. Then it curves due to the $\vec{B}$ field. Ultimately it takes a cycloidal path.

Position of the particle $= (0, y(t), z(t))$ ← To solve for.

Velocity of the particle $= (0, \dot{y}(t), \dot{z}(t))$; $\dot{y}(t) = \frac{dy}{dt}$

We need to calculate the force.

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$
$$= q(E\hat{z} + B\dot{z}\hat{y} - B\dot{y}\hat{z})$$

; Because $\vec{v} \times \vec{B}$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix}$$
$$= B\dot{z}\hat{y} - B\dot{y}\hat{z}$$

So, using $\vec{F} = m\vec{a}$,

$$m(\ddot{y}\hat{y} + \ddot{z}\hat{z}) = q(B\dot{z}\hat{y} + (E - B\dot{y})\hat{z})$$

Separating out the axes:

$$m\ddot{y} = qB\dot{z}\hat{y}$$
$$m\ddot{z} = q(E - B\dot{y})\hat{z}$$

} Coupled ordinary differential equations

Setting $\omega = \frac{qB}{m}$:

$$\ddot{y} = \omega\dot{z}$$
$$\ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$$

} You can solve this by substitution.

$$\dddot{y} = \omega\ddot{z}$$
$$\therefore \dddot{y} = \omega^2\left(\frac{E}{B} - \dot{y}\right)$$
$$\therefore \ddot{v}_y = \omega^2\left(\frac{E}{B} - v_y\right) \ldots \text{etc.}$$

General solution is

$$y(t) = C_1 \cos \omega t + C_2 \sin \omega t + \left(\frac{E}{B}\right)t + C_3$$

$$z(t) = C_2 \cos \omega t - C_1 \sin \omega t + C_4$$

Using boundary conditions that particle started at rest, i.e. at $(0,0,0)$, it had velocity also $(0,0,0)$, we can work out $C_1, \ldots C_4$ and simplify:

$$y(t) = \frac{E}{\omega B}\left(\omega t - \sin(\omega t)\right)$$

$$z(t) = \frac{E}{\omega B}\left(1 - \cos(\omega t)\right)$$

If we let $R = \frac{E}{\omega B}$, and use $\sin^2(\omega t) + \cos^2(\omega t) = 1$, we get:

$$(y - R\omega t)^2 + (z - R)^2 = R^2$$ ← Cycloid!

This is a circle of radius R, centered at $(0, R\omega t, R)$

moving along y-axis at speed ωt.

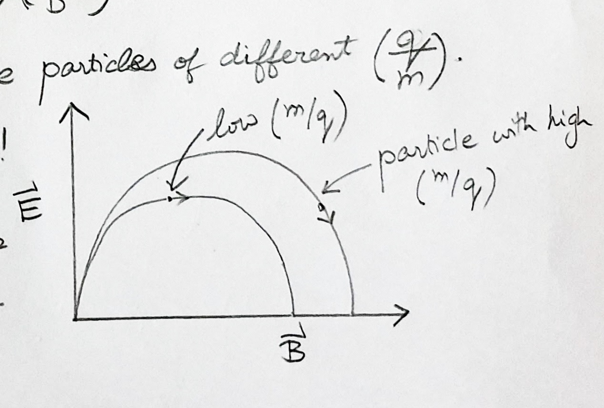
∴ This is a cycloid!

So, what is R?

$$R = \frac{E}{\omega B} = \frac{Em}{qB^2} = \left(\frac{m}{q}\right)\left(\frac{E}{B^2}\right)$$

This means, R can separate particles of different $\left(\frac{q}{m}\right)$.

That is a mass spectrometer!



A muon has charge $-e$, mass $= 106\ \mathrm{MeV}/c^2$

A pion has charge $-e$, mass $= 135\ \mathrm{MeV}/c^2$

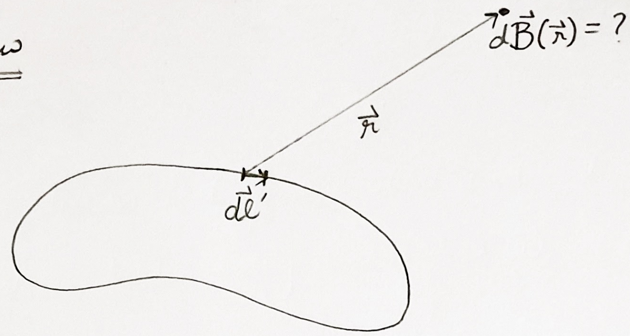
So, a pion will have higher R

and thus it can be separated from muons!

Can we use this to create a beam of muons?

Magnetic field due to a steady current

Biot-Savart's Law



$$d\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \frac{d\vec{l}' \times \hat{r}}{r^2}$$ ← due to the infinitesimal current

$$\therefore \vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l}' \times \hat{r}}{r^2}$$

$$\mu_0 = 4\pi \times 10^{-7}\ N/A^2$$

This is a choice and is used to define Amperes.
Which in turn was used to define Coulombs.

$$\therefore 1\,T = 1\,N/(A\cdot m)$$