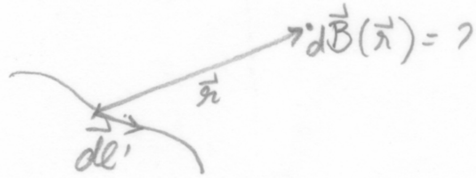
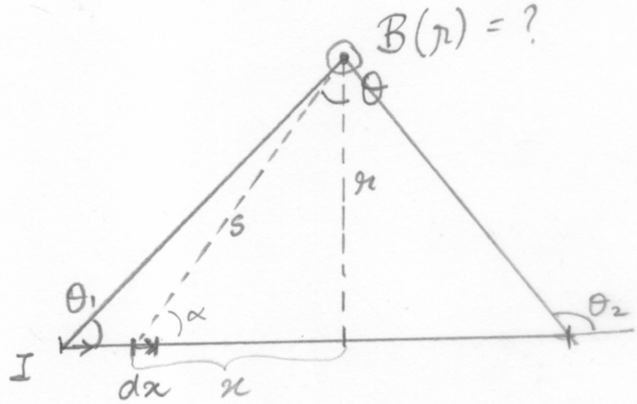
Lecture 18 Magnetostatics Basics Continued

Biot Savart's Law

$$d\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2}$$



Line Current



$$B(r) = \frac{\mu_0 I}{4\pi} \int dx \frac{\sin\alpha}{s^2}$$

$\rightarrow \sin\alpha = \cos\theta$

$x = r\tan\theta$

$\therefore dx = r\sec^2\theta \, d\theta$

$s = \frac{r}{\cos\theta}$

$$\therefore B(r) = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} r\sec^2\theta \, d\theta \cdot \frac{\cos^3\theta}{r^2}$$

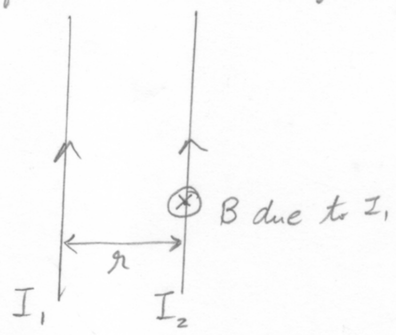
$$= \frac{\mu_0 I}{4\pi r} \int_{\theta_1}^{\theta_2} \cos\theta \, d\theta$$

$$\boxed{B(r) = \frac{\mu_0 I}{4\pi r}\left[\sin\theta_2 - \sin\theta_1\right]}$$

In the case of infinite line current, $\theta_1 \rightarrow \pi/2$, $\theta_2 \rightarrow -\pi/2$

$$\boxed{B = \frac{\mu_0 I}{2\pi r}}$$

So, what would be the force of one infinitely long current on another?



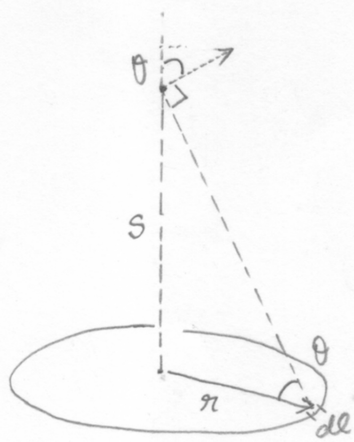
B due to I_1 is $= \dfrac{\mu_0 I_1}{4\pi r}$

Force on I_2 due to B is $= \quad dF = B I_2 \, dl$

$$\boxed{\frac{dF}{dl} = \frac{\mu_0 I_1 I_2}{4\pi r}}$$

So, it is force per unit length of wire

<u>Ring of Current</u>



$$B = \frac{\mu_0 I}{4\pi} \int dl \, \frac{1}{(r^2+s^2)} \cos\theta$$

And $\cos\theta = \dfrac{r}{\sqrt{r^2+s^2}}$

$$\therefore B = \frac{\mu_0 I}{\underset{2}{\cancel{4\pi}}} \frac{r}{(r^2+s^2)^{3/2}} \, \cancel{2\pi} r$$

$$\boxed{B = \frac{\mu_0 I r^2}{2(r^2+s^2)^{3/2}}}$$

For $s = 0$,

$$\boxed{B = \frac{\mu_0 I}{2r}} \text{ constant}$$

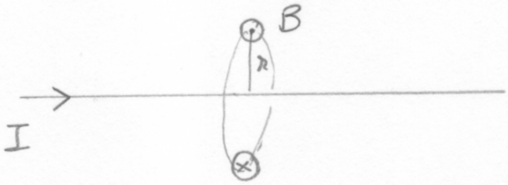
For $s \to \infty$

$$B = \frac{\mu_0 I r^2}{2 s^3 \left(1 + \left(\frac{r}{s}\right)^2\right)^{3/2}}$$

$$\boxed{B = \frac{\mu_0 I r^2}{2 s^3}} \rightarrow \text{falls off as } 1/s^3$$

magnetic dipole!

Infinite line current

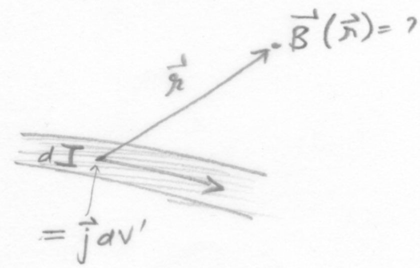


$$B \, 2\pi r = \mu_0 I$$

$$\boxed{B = \frac{\mu_0 I}{2\pi r}}$$

Ampere's Law from Biot-Savart's Law

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{V'} \vec{j}(\vec{r}\,') \times \frac{\hat{r}}{r^2} \, dV'$$



$$\vec{r} = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$$

$$dV' = dx' dy' dz'$$

So,
$$\vec{\nabla} \cdot \vec{B} = \frac{\mu_0}{4\pi} \int_{V'} \vec{\nabla} \cdot \left(\vec{j} \times \frac{\hat{r}}{r^2} \right) dV'$$

Now, $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

So, $\vec{\nabla} \cdot \left(\vec{j} \times \frac{\hat{r}}{r^2} \right) = \left(\frac{\hat{r}}{r^2} \right) \cdot (\vec{\nabla} \times \vec{j}) - \vec{j} \cdot \left(\vec{\nabla} \times \left(\frac{\hat{r}}{r^2} \right) \right)$

Now, $\vec{\nabla} \times \vec{j} = 0$ because differentiation on unprimed variables

$\vec{\nabla} \times \frac{\hat{r}}{r^2} = 0$ as well

So, $\boxed{\vec{\nabla} \cdot \vec{B} = 0}$

And, $$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \frac{\mu_0}{4\pi} \int_{V'} \vec{\nabla} \times \left(\vec{j} \times \frac{\hat{r}}{r^2} \right) dV'$$

Using $$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$$

$$\therefore \; \vec{\nabla} \times \left(\vec{j} \times \frac{\hat{r}}{r^2} \right) = \underbrace{\vec{j} \left(\vec{\nabla} \cdot \frac{\hat{r}}{r^2} \right)}_{4\pi \vec{j}\, \delta^3(\vec{r})} \underbrace{- (\vec{j} \cdot \vec{\nabla}) \left(\frac{\hat{r}}{r^2} \right)}$$

$$-(\vec{j} \cdot \vec{\nabla}) \left(\frac{\hat{r}}{r^2} \right) = (\vec{j} \cdot \vec{\nabla}') \left(\frac{\hat{r}}{r^2} \right)$$ switching integration variable.

$$(\vec{j} \cdot \vec{\nabla}') \left(\frac{x - x'}{r^3} \right) = \vec{\nabla}' \cdot \left[\frac{(x - x')}{r^3} \vec{j} \right] - \left(\frac{x - x'}{r^3} \right) (\vec{\nabla}' \cdot \vec{j})$$

for steady currents, $\vec{\nabla}' \cdot \vec{j} = 0$

So, $$-(\vec{j} \cdot \vec{\nabla}) \frac{\hat{r}}{r^2} \Big|_x = \vec{\nabla}' \cdot \left(\frac{(x - x')}{r^3} \vec{j} \right)$$

This contribution to the integral $\rightarrow$ $$\int \vec{\nabla}' \cdot \left(\left(\frac{x - x'}{r^3} \right) \vec{j} \right) dV'$$

$$= \oint_S \frac{(x - x')}{r^3} \vec{j} \cdot d\vec{A}$$

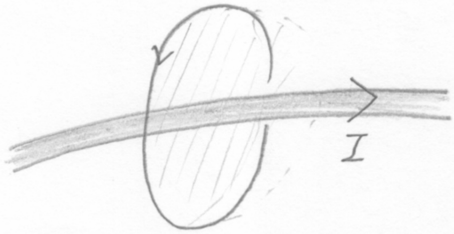
$= 0$ because currents out there at infinity tend to 0.

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \int \vec{j}(\vec{r}')\, 4\pi\, \delta^3(\vec{r} - \vec{r}')\, dV'$$

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}(\vec{r})}$$ $\longleftarrow$ generally true for magnetostatics.

Ampere's Law

The fundamental law of magnetostatics



$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I$$

current that cuts throught **any** surface defined by loop C.

It does not necessarily have to be the simplest surface! This will be important later.

$\oint \vec{B}\cdot d\vec{l} = 0$ for a loop like this

Stokes Law

$$\oint_C \vec{B}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{B})\cdot d\vec{s}$$

Now $\mu_0 I$ can also be expressed as a surface integral over the same surface!

$$\mu_0 I = \mu_0 \int_S \vec{j}\cdot d\vec{s}$$

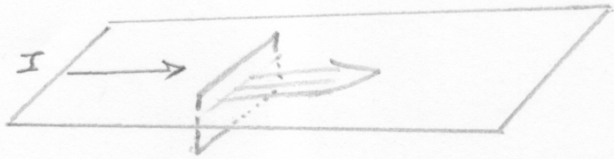
Putting these together,

$$\int_S (\vec{\nabla}\times\vec{B})\cdot d\vec{s} = \mu_0 \int_S \vec{j}\cdot d\vec{s}$$

Since S is arbitrary, $\boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{j}}$

Differential form of Ampere's Law.

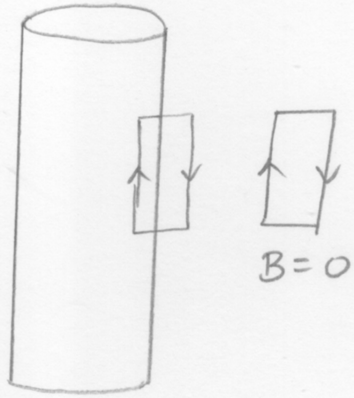
Magnetic field near a surface current



$$2B \cdot l = \mu_0 I = \mu_0 K l$$

$$B = \frac{\mu_0 K}{2} \text{ ; direction as appropriate}$$

Magnetic field inside a solenoid



$$B\,l = \mu_0 n I l \quad ; \; n = \text{turns per length}$$

$$\boxed{B = \mu_0 n I} \leftarrow \text{inside}$$