Lecture 19 . The Magnetic Vector Potential

We can associate a potential to the magnetic field just like we could to the electric field to make some calculations easier. Also sheds light on some fundamental layers of reality.

For electrostatics, we exploited $\vec{\nabla}\times\vec{E} = 0$

to define $\vec{E} = -\vec{\nabla}\Phi$

For magnetics (not necessarily magnetostatics), we exploit $\vec{\nabla}\cdot\vec{B} = 0$

to define $\vec{B} = \vec{\nabla}\times\vec{A}$

Because the divergence of a curl is 0. i.e $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

What are units of $\vec{A}$?

Since $\vec{B} = \vec{\nabla}\times\vec{A} \sim \frac{\partial A}{\partial x}$

$\therefore$ A is like B $\times$ distance.

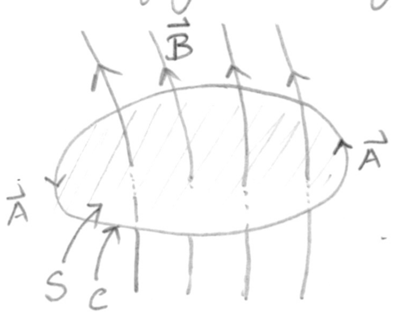
$A \sim Tm = \frac{N}{Am}\cdot m = \frac{N}{A} = \frac{Ns}{C} = \frac{kg\,\frac{m}{s}}{C} = \frac{\text{momentum}}{\text{charge}}$

Just as $\Phi \sim \frac{\text{energy}}{\text{charge}}$ (related to potential energy)

$A \sim \frac{\text{momentum}}{\text{charge}}$ (related to "potential momentum")

What is the equation for $\vec{A}$?

⊙ If you already know $\vec{B}$, you can use $\vec{\nabla}\times\vec{A} = \vec{B}$ to find $\vec{A}$



$$\int_S \vec{\nabla}\times\vec{A}\cdot d\vec{A} = \int_S \vec{B}\cdot d\vec{A}$$

$$\boxed{\oint_C \vec{A}\cdot d\vec{l} = \Upsilon_{flux}}$$

defined as $\int_S \vec{B}\cdot d\vec{A}$

Trying to avoid Φ as the flux.

⊙ To get it from source currents,

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}(\vec{r})$$

So, $$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{j}(\vec{r})$$

Now, $$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$ ⟵ Laplacian of a vector.

$$\therefore \quad \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{j}$$

At this point, we can force the divergence of $\vec{A}$ to be zero without changing its physical implications which only depend on the curl of $\vec{A}$.

We can add any field like $\vec{\nabla}\lambda$ to $\vec{A}$ (because $\vec{\nabla} \times (\vec{\nabla}\lambda) = 0$)

such that: $$\vec{\nabla} \cdot (\vec{A} + \vec{\nabla}\lambda) = 0$$

$$\Rightarrow \quad \nabla^2 \lambda = -\vec{\nabla} \cdot \vec{A}$$

So, if we had a divergence-full $\vec{A}$, we need to find the "compensating" field λ by

$$\lambda = \frac{1}{4\pi} \int \frac{\vec{\nabla} \cdot \vec{A}(\vec{r}')}{|\vec{r}|} dV'$$ [If $\vec{\nabla} \cdot \vec{A}$ went to 0 at ∞]

And add it back to $\vec{A} \rightarrow \vec{A} + \vec{\nabla}\lambda$. Thus you could force $\vec{\nabla} \cdot \vec{A} = 0$

This is similar to how in electrostatics, we had a freedom to change

$$\Phi \rightarrow \Phi + \text{constant}$$

And this would not change the physics.

$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\lambda$ would not change the physics

PRINCIPLE OF GAUGE INVARIANCE

Our choice of convenience to let $\vec{\nabla} \cdot \vec{A} = 0$ is called Coulomb Gauge.

Through this choice, we write

$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{j}}$$

This is EXACTLY Poisson's Equation but for each coordinate independently.

Poisson's Equation for the Magnetic Vector Potential

Gets the exact same treatment as Poisson's for electrostatics. $\nabla^2 \vec{A} = -\mu_0 \vec{j}$

We can find a Green's function:

$$\nabla^2 G_x(\vec{r}, \vec{r}') = -\mu_0 \delta^3(\vec{r} - \vec{r}') \qquad \text{could solve } \nabla^2 \vec{G} =$$

And then we can use this Green's function to find $\vec{A}$ as

$$A_x(\vec{r}) = \int_V j_x(\vec{r}')\, G_x(\vec{r}, \vec{r}')\, dV' - \epsilon_0 \int_S A_x(\vec{r}') \frac{\partial G_x}{\partial \eta}\, dS' + \epsilon_0 \int_S G_x(\vec{r}', \vec{r}) \frac{\partial A_x}{\partial \eta}\, dS'$$

Assuming $\vec{A}$ and $\frac{\partial \vec{A}}{\partial \eta}$ go to zero at infinity,

$$G_x(\vec{r}, \vec{r}') = \frac{\mu_0}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|}$$

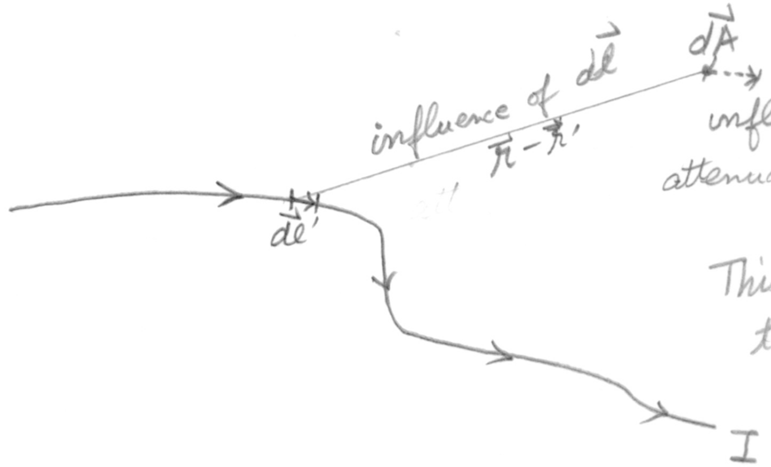
So,

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{j}(\vec{r}')\, dV'}{|\vec{r} - \vec{r}'|}}$$

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{I(\vec{r}')\, d\vec{l}'}{|\vec{r} - \vec{r}'|}}$$

"Coulomb's Law" or Green's function for the vector potential

Intuition

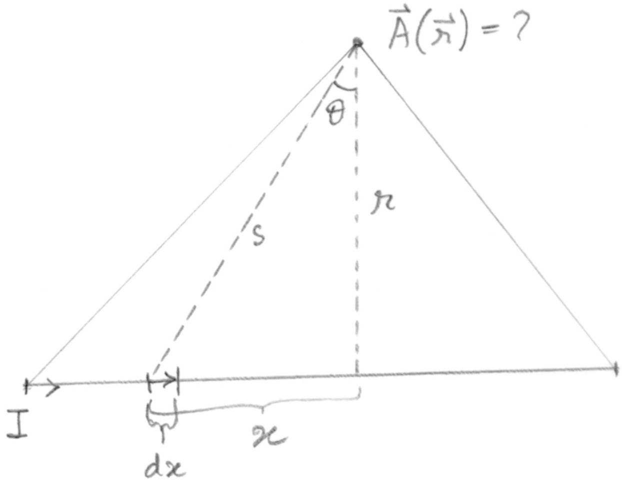


influence of $d\vec{l}$ in direction of $d\vec{l}$ attenuated by $\frac{1}{|\vec{r} - \vec{r}'|}$ from $d\vec{l}$

This has to be integrated over the entire wire.

Example

Line Current (Done with Biot-Savart's Law in Lecture 18)



$\vec{A}$ will all be in the $\vec{x}$ direction

So, $A = \frac{\mu_0}{4\pi} I \int \frac{dx}{s}$, $x = r\tan\theta$

$\therefore dx = r\sec^2\theta \, d\theta$

& $s = r\sec\theta$

$= \frac{\mu_0 I}{4\pi} \int \frac{r\sec^2\theta \, d\theta}{r\sec\theta}$

$A = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \sec\theta \, d\theta$

So, $\boxed{A = \frac{\mu_0 I}{4\pi} \ln\left(\frac{\sec\theta_2 + \tan\theta_2}{\sec\theta_1 + \tan\theta_1}\right)}$

From this, can we derive $\vec{B}$? $\vec{B} = \vec{\nabla} \times \vec{A}$

Only $\hat{\phi}$ component around wire is relevant

So, $(\vec{\nabla} \times \vec{A})_{\hat{\phi}} = -\frac{\partial A}{\partial r}\hat{\phi}$ [Curl in cylindrical coordinates]

$\therefore (\vec{\nabla} \times \vec{A})_{\hat{\phi}} = -\frac{\mu_0 I}{4\pi} \frac{\partial}{\partial r}\left[\ln\left(\frac{\sqrt{r^2+x_2^2}+x_2}{\cancel{r}}\right) - \ln\left(\frac{\sqrt{r^2+x_1^2}+x_1^2}{\cancel{r}}\right)\right]$

because $\sec\theta = \frac{\sqrt{r^2+x^2}}{r}$

$\tan\theta = \frac{x}{r}$

$= \frac{-\mu_0 I}{4\pi}\left[\frac{\cancel{2}r}{\{x_2+\sqrt{r^2+x_2^2}\}\cancel{2}} \frac{1}{\sqrt{r^2+x_2^2}} - \frac{\cancel{2}r}{\{x_1+\sqrt{r^2+x_1^2}\}\cancel{2}} \frac{1}{\sqrt{r^2+x_1^2}}\right]\hat{\phi}$

$= \frac{-\mu_0 I r}{4\pi}\left[\frac{1}{\{x_2+\sqrt{r^2+x_2^2}\}\sqrt{r^2+x_2^2}} - \frac{1}{\{x_1+\sqrt{r^2+x_2^2}\}\sqrt{r^2+x_1^2}}\right]\hat{\phi}$

This can be simplified ..

$$B_{\hat{\phi}} = (\vec{\nabla}\times\vec{A})_{\hat{\phi}} = \frac{-\mu_0 I r}{4\pi}\left[\frac{x_2-\sqrt{r^2+x_2^2}}{\{x_2^2-(r^2+x_2^2)\}\sqrt{r^2+x_2^2}} - \frac{x_1-\sqrt{r^2+x_1^2}}{\{x_1^2-(r^2+x_1^2)\}\sqrt{r^2+x_1^2}}\right]\hat{\phi}$$

$$= \frac{+\mu_0 I r}{4\pi}\left[\frac{x_2-\sqrt{r^2+x_2^2}}{r^2\sqrt{r^2+x_2^2}} - \frac{x_1-\sqrt{r^2+x_1^2}}{r^2\sqrt{r^2+x_1^2}}\right]\hat{\phi}$$

$$= \frac{\mu_0 I}{4\pi r}\left[\frac{x_2}{\sqrt{r^2+x_2^2}} - \cancel{1} - \frac{x_1}{\sqrt{r^2+x_1^2}} + \cancel{1}\right]\hat{\phi}$$

$$\boxed{B_\phi = \frac{\mu_0 I}{4\pi r}\left[\sin\theta_2 - \sin\theta_1\right]}$$

↳ Which is what we obtained in Lecture 18 much more simply using Biot-Savart's Law.

So, $\vec{A}$ ~~is~~ is not always practically helpful

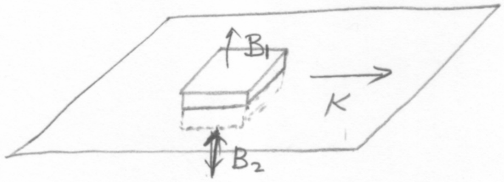
But it does shed light on fundamental physics

↳ Has implications in Quantum Mechanics.

"Aharanov - Bohm effect"

Continuity Conditions

$\vec{B}$ & $\vec{A}$ across surface currents



$$\vec{\nabla} \vec{B} = 0 \Rightarrow B_1^{\perp} = B_2^{\perp}$$

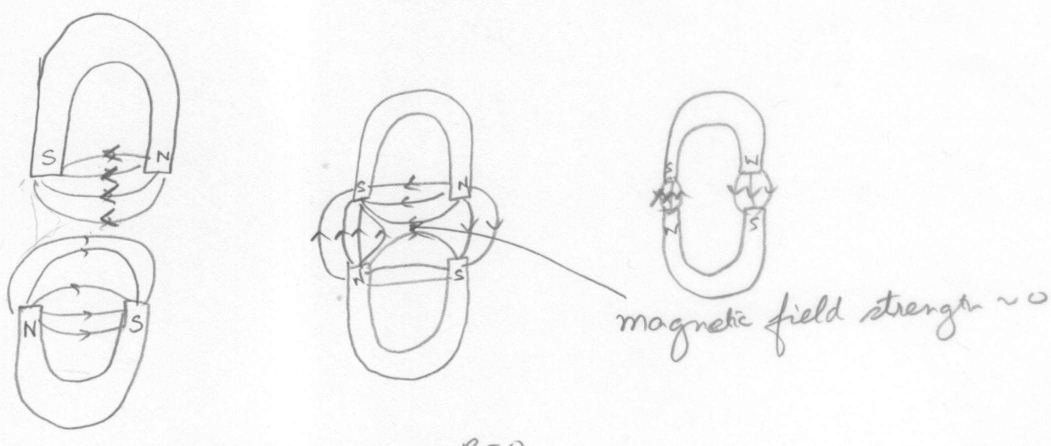
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \Rightarrow B_1^{\parallel} - B_2^{\parallel} = \mu_0 K$$

$$\vec{\nabla} \cdot \vec{A} = 0 \text{ (Coulomb gauge)} \Rightarrow A_1^{\perp} = A_2^{\perp}$$

$$\vec{\nabla} \times \vec{A} = \vec{B} \Rightarrow A_1^{\parallel} - A_2^{\parallel} = \text{flux enclosed} = 0$$

$$\text{but, } \frac{\partial \vec{A}_1}{\partial n} - \frac{\partial \vec{A}_2}{\partial n} = -\mu_0 \vec{K}$$

Magnetic Reconnection



Reconnection happens near $B = 0$

Show that $h = \int dV \, \vec{A} \cdot \vec{B}$ $= 0$ for disconnected flux tubes

$= 2 \Phi_1 \Phi_2$ for linked tubes.