Lecture 2

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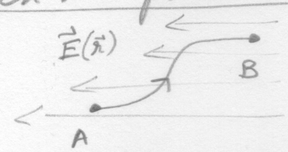
Relationship between the potential and work done



What work would you have to do to bring charge q from $\vec{A}$ to $\vec{B}$?

$$W = -\int_{\vec{A}}^{\vec{B}} \underbrace{q\vec{E}}_{\vec{F}} \cdot d\vec{r}$$

Force on charge is $\vec{F} = q\vec{E}$

You exert $\vec{F}' = -\vec{F}$

Now, $\vec{E} = -\vec{\nabla} V$

$$\therefore \quad W = q\left[V(\vec{B}) - V(\vec{A})\right]$$

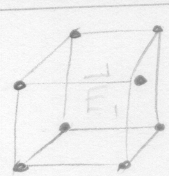
This is path-independant. Because $\vec{E} = -\vec{\nabla} V$ and therefore $\vec{\nabla} \times \vec{E} = 0$.

This is called a conservative field.

> Is the $\vec{E}$ field always conservative?
No!

~~Matching conditions for $\vec{E}$ and ϕ across a surface charge~~

Earnshaw's Theorem

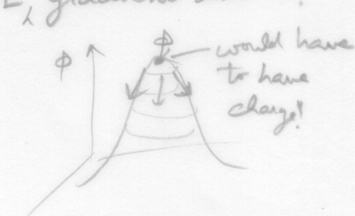


STATEMENT: The ϕ in a charge-free region of space has its maximum or minimum at boundary.

If there were a local minimum or maximum, there would be an $\vec{E}$, i.e. gradient around that point. If there is a $\vec{E}$ around a point, then, there must be charge there!

$\vec{E} = -\vec{\nabla}\phi$

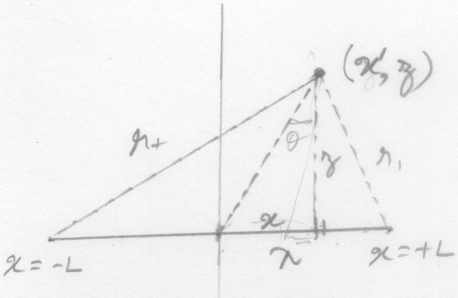
$$\oint_S \vec{E} \cdot d\vec{s} > 0$$



So, you cannot confine plasma electrostatically.

Or at long as you can write $\vec{E} = -\vec{\nabla}\phi$. Breaks down with $\frac{\partial \vec{B}}{\partial t}$.

⊙ Equipotentials are lines of equal potential. What are the equipotential lines around a line charge?



$$\phi(x',z') = \frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{L}\frac{dx}{\sqrt{(x'-x)^2+z^2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\int_{\theta_1}^{\theta_2}\frac{d\theta}{\cos\theta}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{\overbrace{\sqrt{(L-x')^2+z^2}}^{r_1}+(L-x')}{\underbrace{\sqrt{(L+x')^2+z^2}}_{r_2}-(L+x')}\right]$$

Let $u = \frac{1}{2}(r_+ + r_-)$

$t = \frac{1}{2}(r_- - r_+)$

$$\therefore \phi(x',z') = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{u+t+L-z}{u-t-L-z}\right)$$

Using $ut = -zL \longrightarrow \left[\frac{(u+L)(1+t/L)}{(u-L)(1+t/L)}\right]$

$$\therefore \phi = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{u+L}{u-L}\right)$$

$\therefore$ ϕ is lines of constant $u = r_+ + r_-$

$\therefore$ Ellipses.

$t^2 = (x'-x)^2$

and

$x'-x = z\tan\theta$

or

$x - x' = z\tan\theta$

$dx = z\sec^2\theta\, d\theta$

$\frac{\sec^2\theta\, d\theta}{z\sec\theta}$

$\int \sec\theta\, d\theta = \ln|\sec\theta+\tan\theta|$

$\frac{r_- + L - x'}{r_+ - L - x'}$

$\frac{r_1 + L - x'}{r_2 - L - x'}$

r_1

$\frac{r_1 - x' + L}{r_2 - x' - L}$

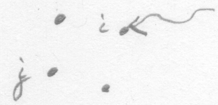
$\frac{1}{2}r_+ + \frac{r_-}{2} + L$

$r_1 - x' =$

$u + t + L - z = (u+L)(1+t/L)$

$u + L + \cancel{\frac{ut}{L}} + t$

Energy required to construct a group of charges



$$W = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{N} \sum_{i=1}^{i>j} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

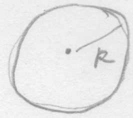
$$= \frac{1}{4\pi\epsilon_0} \frac{1}{2} \sum_{j=1}^{N} \sum_{i=1}^{N} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$; $\frac{1}{2}$ accounts for the double-counting

$$U_E = W = \frac{1}{2} \sum_{i=1}^{N} q_i \phi(\vec{r}_i)$$

Similarly, for a continuous distribution,

$$U_E = \frac{1}{2} \int \rho \phi \, dV$$

> Energy of a thin shell



$$W = \frac{1}{2} \int \sigma V \, da$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{R} \cdot \frac{1}{2} \cdot \underbrace{4\pi R^2 \sigma}_{\to q}$$

$$\boxed{W = \frac{q^2}{8\pi\epsilon_0 R}}$$

Only in terms of the electric field

$$U_E = \frac{1}{2} \int \rho \phi \, dV$$

But $\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$

$$\therefore \; U_E = \frac{\epsilon_0}{2} \int (\vec{\nabla} \cdot \vec{E}) \phi \, dV$$

Because

$$\vec{\nabla} \cdot (f\vec{A}) = \vec{A} \cdot \vec{\nabla} f + f \vec{\nabla} \cdot \vec{A}$$

$$\therefore \; f \vec{\nabla} \cdot \vec{A} = \vec{A} \cdot \vec{\nabla} f - \vec{\nabla} \cdot (f\vec{A})$$

$$\int f (\vec{\nabla} \cdot \vec{A}) dV = \int \vec{A} \cdot (\vec{\nabla} f) dV - \int \vec{\nabla} \cdot (f\vec{A}) dV$$

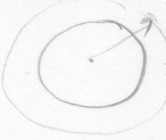
Integration by parts:

$$U_E = \frac{\epsilon_0}{2} \left[\int \vec{\nabla} \cdot (\phi \vec{E}) dV - \int \vec{E} \cdot \vec{\nabla} \phi \, dV \right]$$

$$= \frac{\epsilon_0}{2} \left[\underbrace{\oint_S \phi \vec{E} \cdot d\vec{s}}_{\to 0 \text{ at } \infty} + \int E^2 dV \right]$$

$$\therefore \; \boxed{U_E = \frac{\epsilon_0}{2} \int E^2 dV}$$

So, what is the potential energy of a charged shell?



Inside, $E = 0$

Outside, $E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$

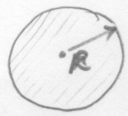
So, $E^2 = \frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4}$

$$\therefore U_E = \frac{\epsilon_0}{2}\int_R^\infty \frac{q^2}{(4\pi\epsilon_0)^2} r^{-4}\, dr \cdot 4\pi r^2$$

$$\boxed{U_E = \frac{q^2}{8\pi\epsilon_0 R}}$$

Could the mass of an electron be electrostatic in origin?

$c^2 = \frac{1}{\mu_0 \epsilon_0}$



Electron is a thin shell of radius R

$$U = \frac{1}{2}\cdot\frac{e^2}{4\pi\epsilon_0 R} \longrightarrow = m_e c^2$$

What is R = ?

$$R = \frac{e^2}{8\pi \epsilon_0 m_e \underbrace{c^2}_{\frac{1}{\mu_0\epsilon_0}}} = \frac{e^2\mu_0}{8\pi m_e} =$$

$e = 1.6 \times 10^{-19} C$

$\mu_0 = 4\pi \times 10^{-7}\ N/A^2$

$m_e = 9.1 \times 10^{-31} kg$

$$\therefore R = \frac{(1.6\times 10^{-19})^2 \cancel{C^2} \times \cancel{4\pi}\times 10^{-7}\ \cancel{N}/\cancel{A^2}\ \frac{kg\, m}{sec^2}}{2\,\cancel{8\pi} \times 9.1\times 10^{-31}\ kg\ \cancel{C^2}}$$

$= 0.14 \times 10^{-14}\ m$

$= 1.4 \times 10^{-15}\ m$

classical radius

Around the scale at which renormalization and loop diagrams become important in QED.

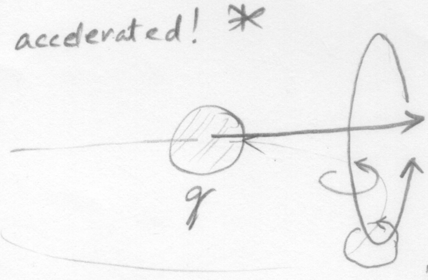
Where e^+e^- of the vacuum become relevant!

$\cancel{38}$ 45 −31 14

* But this mass is inconsistent with braking from electrodynamics if accelerated! *

$\nabla \cdot B = \mu_0 J$

$\nabla \times E = -\frac{\partial B}{\partial t}$



F = ma

analogy gives

$$m_e = \left(\frac{2}{3}\right)\frac{e^2}{4\pi\epsilon_0 R}\frac{1}{c^2}$$

not 1/2. This caused a lot of confusion.

Led Feynman to QED. Read Nobel lecture!