Lecture 20 — Multipole Expansion of the Vector Potential

We found the "Green's function" form of the vector potential

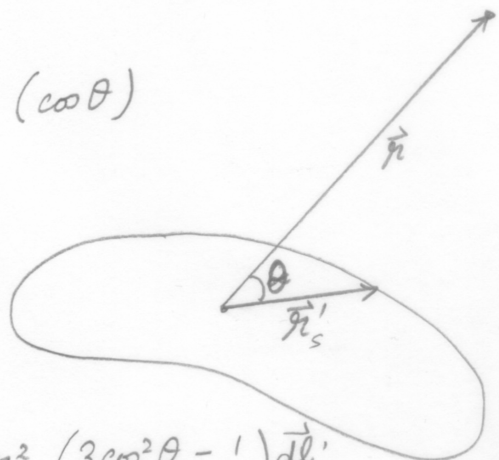
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{r}')\,dV'}{|\vec{r}-\vec{r}'|}$$

or

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\oint \frac{I(\vec{r}')\,d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

As we saw in Lecture 3, $\frac{1}{|\vec{r}-\vec{r}'|}$ can be expanded in terms of r' as follows:

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta)$$



So,

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\Bigg[\underbrace{\frac{1}{r}\oint d\vec{l}'}_{0} + \frac{1}{r^2}\oint r_s\cos\theta\, d\vec{l}' + \frac{1}{r^3}\oint r_s^2\left(\frac{3\cos^2\theta-1}{2}\right)d\vec{l}' + \frac{1}{r^4}\oint r_s^3\left(\frac{5\cos^3\theta-3\cos\theta}{2}\right)d\vec{l}'\Bigg]$$

0 This is a consequence of $\vec{\nabla}\cdot\vec{B}=0$

Break it down

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r_s\cos\theta\, d\vec{l}' \quad \text{. Dipole term}$$

$$+ \frac{\mu_0 I}{4\pi r^3}\oint r_s^2\left(\frac{3\cos^2\theta-1}{2}\right)d\vec{l}' \quad \text{.. Quadrupole term}$$

$$+ \frac{\mu_0 I}{4\pi r^4}\oint r_s^3\left(\frac{5\cos^3\theta-3\cos\theta}{2}\right)d\vec{l}' \quad \text{. Octopole term}$$

Dipole term is the dominant term

$$\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r_s \cos\theta \, d\vec{l}$$

$$= \frac{\mu_0 I}{4\pi r^2}\oint (\hat{r}\cdot\vec{r}_s)\, d\vec{l}$$

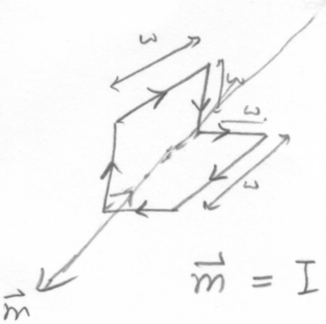
Now, $\oint (\hat{r}\cdot\vec{r}_s)\, d\vec{l} = -\hat{r} \times \int d\vec{a}'$

So, $\vec{A}_{dipole}(\vec{r}) = \dfrac{\mu_0}{4\pi}\left[I\int d\vec{a}\right] \times \dfrac{\hat{r}}{r^2}$

$$\boxed{\vec{A}_{dipole} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}} \; ; \quad \boxed{\vec{m} = I\int d\vec{a} = I\vec{a}}$$

$\hookrightarrow$ vector area of the loop

Example



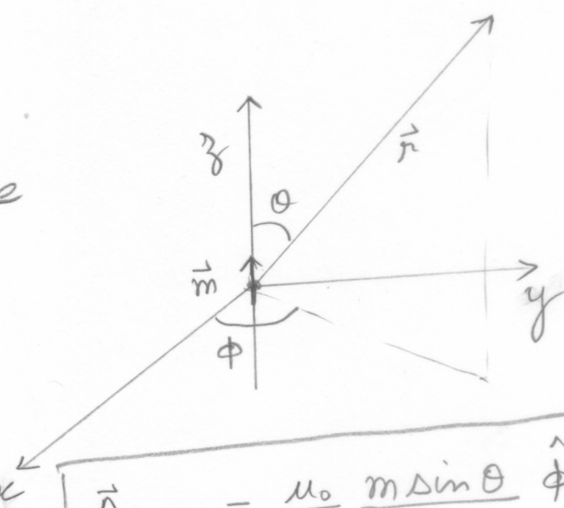
$$\vec{m} = I\omega^2\hat{y} + I\omega^2\hat{z}$$

Ideal Magnetic dipole

$I \to \infty$

$a \to 0$

Such that m is finite



$$\boxed{\vec{B}_{dipole} = \frac{\mu_0 I}{4\pi r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]}$$

$$\boxed{\vec{A}_{dipole} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}}$$

$$\boxed{\vec{B}_{dipole} = \vec{\nabla}\times\vec{A}_{dipole} = \frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]}$$