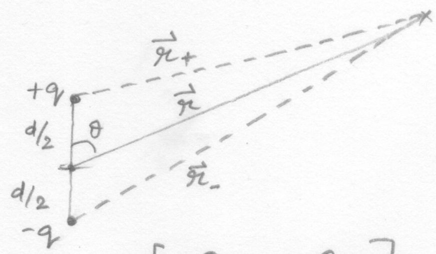
## Lecture 3

Why are electrostatic multipoles interesting?

Because most macroscopic things are like that.

### A simple dipole




$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r_+} - \frac{q}{r_-}\right]$$

Now, $r_+ = r^2 + \left(\frac{d}{2}\right)^2 - 2r\left(\frac{d}{2}\right)\cos\theta$

& $r_- = r^2 + \left(\frac{d}{2}\right)^2 + 2r\left(\frac{d}{2}\right)\cos\theta$

$\therefore \frac{1}{r_+} = \frac{1}{r}\left(1 + \left(\frac{d}{2r}\right)^2 - \left(\frac{d}{r}\right)\cos\theta\right)^{-1/2} \simeq \frac{1}{r}\left(1 - \frac{d}{2r}\cos\theta\right)$

& $\frac{1}{r_-} = \frac{1}{r}\left(1 + \left(\frac{d}{2r}\right)^2 + \left(\frac{d}{r}\right)\cos\theta\right)^{-1/2} \simeq \frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta\right)$

$$\therefore \boxed{\phi(\vec{r}) = \frac{q\,d\cos\theta}{4\pi\epsilon_0 r^2}} \longrightarrow \phi(\vec{r}) = \frac{q\,\hat{r}\cdot\vec{d}}{4\pi\epsilon_0 r^2}$$

$$\boxed{\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}}$$ where $\vec{p} = q\vec{d}$, for $r \gg d$

This happens to be generally true.

### So, what is the $\vec{E}$ field?

$$\vec{E} = -\vec{\nabla}\phi$$

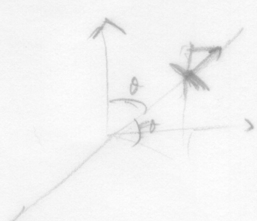
$$\therefore \vec{E} = \frac{-1}{4\pi\epsilon_0}\vec{\nabla}\left(\frac{\vec{p}\cdot\hat{r}}{r^2}\right)$$

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{r}(\hat{r}\cdot\vec{p}) - \vec{p}}{r^3}\right]}$$ for $r \gg d$

In spherical coordinates,

taking, $\vec{p} = p\hat{z}$

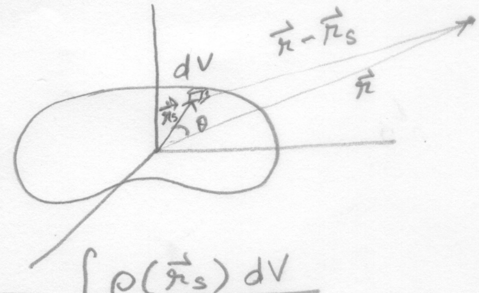
& $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$




$$\boxed{\vec{E}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]}$$

Dipole moments are vectors and can add.

# Multipole moments from a distribution of charges




$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}_s)\, dV}{|\vec{r}-\vec{r}_s|}$$

Now, $|\vec{r}-\vec{r}_s| = \sqrt{r^2 + r_s^2 - 2 r r_s \cos\theta} = r\sqrt{1 + \left(\frac{r_s}{r}\right)^2 - 2\left(\frac{r_s}{r}\right)\cos\theta}$

$$|\vec{r}-\vec{r}_s| = r\left(1 - 2\left(\frac{r_s}{r}\right)\cos\theta + \left(\frac{r_s}{r}\right)^2\right)^{1/2}$$

$$\therefore \frac{1}{|\vec{r}-\vec{r}_s|} = \frac{1}{r}\left(1 - 2\left(\frac{r_s}{r}\right)\cos\theta + \left(\frac{r_s}{r}\right)^2\right)^{-1/2}$$

$$= \frac{1}{r}\left(1 + \left(\frac{r_s}{r}\right)\cos\theta - \frac{1}{2}\left(\frac{r_s}{r}\right)^2 + \frac{3}{8}\left\{\left(\frac{r_s}{r}\right)^2\left(\frac{r_s}{r} - 2\cos\theta\right)^2\right\}\right.$$

$$\left. - \frac{5}{16}\left\{\left(\frac{r_s}{r}\right)^3\left(\frac{r_s}{r} - 2\cos\theta\right)^3\right\} \cdots\right.$$

$$(1+\epsilon)^{-1/2} = 1 - \frac{\epsilon}{2} + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3$$

Combining terms of the same orders,

$$\frac{1}{|\vec{r}-\vec{r}_s|} = \frac{1}{r}\left[1 + \left(\frac{r_s}{r}\right)\cos\theta + \left(\frac{r_s}{r}\right)^2\left(\frac{3\cos^2\theta - 1}{2}\right) + \left(\frac{r_s}{r}\right)^3\left(\frac{5\cos^3\theta - 3\cos\theta}{2}\right) + \cdots\right]$$

$$\therefore \frac{1}{|\vec{r}-\vec{r}_s|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r_s}{r}\right)^n \underbrace{P_n(\cos\theta)}_{\text{Legendre Polynomials.}}$$

We can align $\vec{r}$ with $\hat{z}$ and then $\theta$ is the polar angle.

$$\boxed{\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r_s^n P_n(\cos\theta)\,\rho(\vec{r}_s)\, dV_s}$$

Multipole expansion of $\phi$ in powers of $1/r$

This is EXACT

So, qualitatively, $\phi$ of monopole $\sim 1/r$

of dipole $\sim 1/r^2$

of quadrupole $\sim 1/r^3$

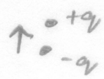
of octopole $\sim 1/r^4$

## Break it down

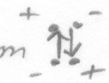
$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\vec{r}_s)\, dV_s \quad \text{---- Monopole term.}$$

Note, $r$ is a constant. Not integrated over.

$$+ \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int \rho(\vec{r}_s)\, r_s \cos\theta \, dV_s \quad \text{--- Dipole term}$$

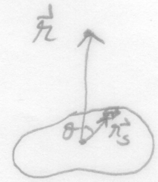



$$+ \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int \rho(\vec{r}_s)\, r_s^2 \left(\frac{3\cos^2\theta - 1}{2}\right) dV_s \quad \text{--- Quadrupole term}$$



$$+ \frac{1}{4\pi\epsilon_0}\frac{1}{r^4}\int \rho(\vec{r}_s)\, r_s^3 \left(\frac{5\cos^3\theta - 3\cos\theta}{2}\right) dV_s \quad \text{--- Octopole term}$$

What is the dipole moment of a charge distribution?




It comes from the second term.

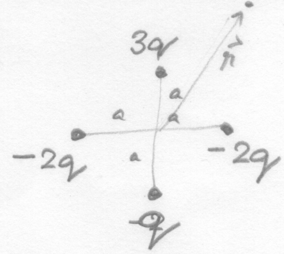
$$\phi(\vec{r})_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int \rho(\vec{r}_s)\, r_s \cos\theta \, dV_s$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot}{r^2}\underbrace{\int \rho(\vec{r}_s)\, \vec{r}_s \, dV_s}_{\vec{p}\ \text{the dipole moment}}$$

Then, we can write

$$\boxed{\phi(\vec{r})_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}}$$

*Example*



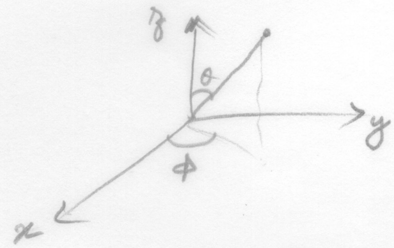

Total charge: $Q = 0$ ; monopole $= 0$

Dipole $= \vec{p} = (3qa - qa)\hat{z} + (-2qa + 2qa)\hat{x}$

$= 2qa\hat{z}$

$\therefore \quad \vec{p}\cdot\hat{r} = 2qa\cos\theta$

$$\therefore \quad \boxed{\phi = \frac{1}{4\pi\epsilon_0}\frac{2qa\cos\theta}{r^2}}$$




## Spherical Charge Distributions

Deserve treatment in spherical coordinates.

$$\nabla^2\phi = 0$$

is
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

Assuming azimuthal symmetry, i.e. $V$ is independent of $\phi$,

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$$

We look for solutions that are products:

$$V(r,\theta) = R(r)\eta(\theta)$$

Putting this in,

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial(R\eta)}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial(R\eta)}{\partial\theta}\right) = 0$$

$$\therefore \quad \eta\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\eta}{\partial\theta}\right) = 0$$

Dividing throughout by $V = R\eta$,

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\eta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\eta}{\partial\theta}\right) = 0$$

We can separate these two as

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = l(l+1) \quad ; \quad \frac{1}{\eta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\eta}{\partial\theta}\right) = -l(l+1)$$

Look at the radial equation,

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = l(l+1)$$

Solution is of the form:

$$R(r) = Ar^l + \frac{B}{r^{l+1}}$$

$$\frac{\partial R}{\partial r} = Alr^{l-1} - (l+1)Br^{-l-2}$$

$$r^2\frac{\partial R}{\partial r} = Alr^{l+1} - (l+1)B^{-l}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = Al(l+1)r^{l} + (l+1)lB^{-l-1}$$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = \frac{Al(l+1)r^l + l(l+1)B^{-l-1}}{Ar^l + Br^{l-1}} = l(l+1)$$

And angular solution:

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\eta}{d\theta}\right) = -l(l+1)\sin\theta \cdot \eta$$

$$\eta(\theta) = P_l(\cos\theta)$$

Legendre polynomials

$$\longrightarrow P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$

Rodrigues formula.

Other solution blows up at $\theta = 0$, $\theta = \pi$.

So, the most general solution for $\Phi$ with azimuthal symmetry is

$$\boxed{\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)}$$

$A_l r^l$ → inside the charge distribution

$\frac{B_l}{r^{l+1}}$ → outside

Legendre polynomials are orthogonal

$$\int_{-1}^{1} P_l(x)\,P_{l'}(x)\,dx = \begin{cases} 0 & \text{if } l \neq l' \\ \frac{2}{2l+1} & \text{if } l = l' \end{cases}$$

$$A_l = \int dV_s \frac{1}{r_s^{l+1}}\rho(\vec{r}_s)P_l(\cos\theta)$$

$$B_l = \int dV_s\, r_s^l\,\rho(\vec{r}_s)P_l(\cos\theta)$$