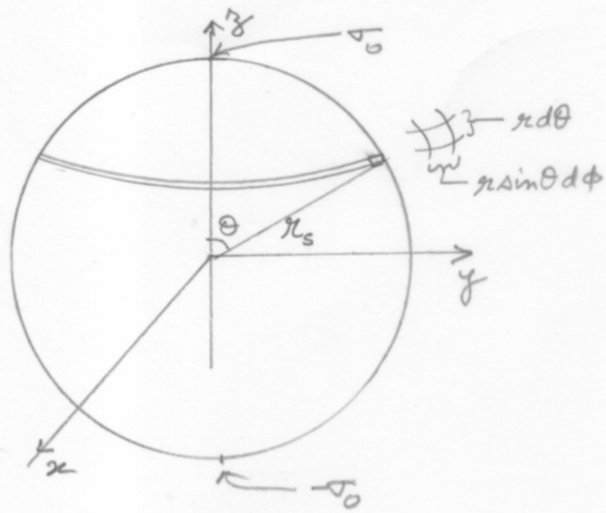
Lecture 4

Example



Compute the dipole moment of this shell with $\sigma = \sigma_0 \cos\theta$ wrt the center.

From, $\vec{p} = \int \rho(\vec{r}_s)\, \vec{r}_s \, dV_s$

We write:

$$p_z = \int \sigma_0 \cos\theta \,.\, r_s \cos\theta \,.\, r_s^2 \sin\theta \, d\theta \, d\phi$$

$$p_z = \sigma_0 r_s^3 \cdot 2\pi \int_0^{\pi} \sin\theta \cos^2\theta \, d\theta$$

Let $t = \cos\theta$ $\therefore$ $p_z = 2\pi \sigma_0 r_s^3 \int_{-1}^{1} t^2 dt$

$\therefore dt = -\sin\theta d\theta$

$$\boxed{p_z = \frac{4\pi}{3} \sigma_0 r_s^3}$$

What is the ϕ at $\vec{r}$?

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2}$$

$$= \frac{1}{\cancel{4\pi}\epsilon_0} \frac{\cancel{4\pi}}{3} \sigma_0 r_s^3 \cdot \frac{\cos\theta}{r^2}$$

$$\phi(\vec{r}) = \frac{\sigma}{\epsilon_0} \begin{cases} \dfrac{r_s^3}{r^2} \cos\theta & \text{outside the shell} \\ r \cos\theta & \text{inside the shell} \end{cases}$$

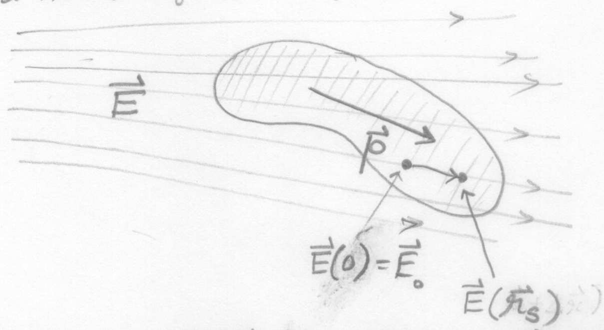
- What is the $\vec{E}$ outside and inside the sphere?

Force on a dipole

Suppose a body has a dipole moment, but no other moment. Total charge = 0

Clearly, a uniform $\vec{E}$ field will not move it.

There must be a non-uniform $\vec{E}$ field to move it.



We can expand the non-uniform $\vec{E}$ field thus around a constant $\vec{E}(\vec{r})=\vec{E}_0$: $\vec{E}(\vec{r}_s) = \vec{E}(0) + (\vec{r}_s\cdot\vec{\nabla})\vec{E}(\vec{r}) + \cdots$

$$\therefore\ \vec{F} = \int dV_s\, \rho(\vec{r}_s)\, \vec{E}(\vec{r}_s)$$

$$= \int dV_s\, \rho(\vec{r}_s)\left[\vec{E}_0 + (\vec{r}_s\cdot\vec{\nabla})\vec{E}(\vec{r})\right]$$

First term goes to zero. So,

$$\vec{F} = \int dV_s\, \rho_s(\vec{r}_s)\,(\vec{r}_s\cdot\vec{\nabla})\vec{E}(0)$$

Now,

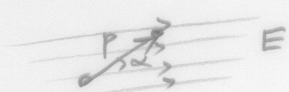
$$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}_0$$

Because: $\vec{\nabla}(\vec{p}\cdot\vec{E}) = \vec{p}\times\cancel{(\vec{\nabla}\times\vec{E})} + \vec{E}\times\cancel{(\vec{\nabla}\times\vec{p})} + (\vec{p}\cdot\vec{\nabla})\vec{E} + \cancel{(\vec{E}\cdot\vec{\nabla})\vec{p}}$

$$\boxed{\vec{F} = \vec{\nabla}(\vec{p}\cdot\vec{E})}$$

Potential Energy $U_E = -\int \vec{F}\cdot d\vec{r} = -\int \vec{\nabla}(\vec{p}\cdot\vec{E})\, dr = -\vec{p}\cdot E$

$$\boxed{U_E = -\vec{p}\cdot\vec{E}}$$



Torque on a dipole

$$\delta\vec{p} = \delta\vec{\alpha}\times\vec{p}$$

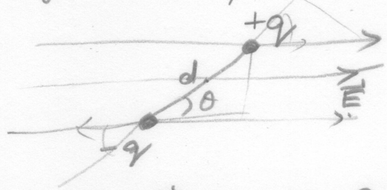
$$\frac{\partial U_E}{\partial\alpha} = -\frac{\partial\vec{p}}{\partial\alpha}\cdot\vec{E}$$

$$= -(\hat{\alpha}\times\vec{p})\cdot\vec{E}$$

$$\frac{\partial U_E}{\partial\alpha} = -(\vec{p}\times\vec{E})\cdot\hat{\alpha}$$

$$\therefore\ \boxed{\vec{\tau} = -\vec{p}\times\vec{E}}$$

Does this make sense for a simple dipole?



Net force $= \vec{F}_x = q\,d\cos\theta \cdot \dfrac{dE}{dx}$; $\vec{F} = q(\vec{d}\cdot\vec{\nabla})\vec{E}$

$= \dfrac{d}{dx}\left(q\,d\cos\theta\,E\right)$ $\quad = (\vec{p}\cdot\vec{\nabla})\vec{E}$

$\boxed{F_x = \dfrac{d}{dx}(\vec{p}\cdot\vec{E})} \rightarrow \boxed{\vec{F} = \vec{\nabla}(\vec{p}\cdot\vec{E})}$

Torque

$\tau = qE\sin\theta \cdot d$

$\boxed{\vec{\tau} = \vec{p}\times\vec{E}}$