Lecture 5

An ideal conductor allows the free movement of charges.

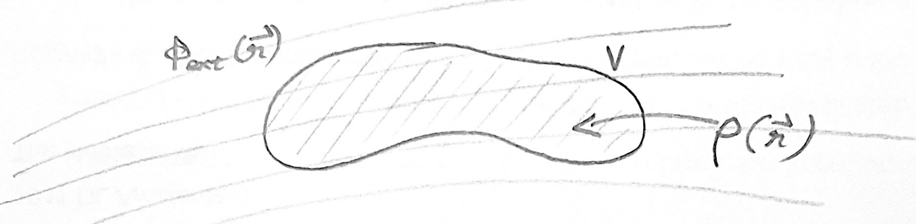
In metals, electrons in conduction band.

In ionic solutions, it is the ions.

$E = 0$, $\rho = 0$ inside the conductor.

Thomson's Theorem of Electrostatics

The electrostatic energy of a body of fixed shape and size is minimized when its charge Q distributes itself such that the potential ϕ is constant throughout the body. Since $\vec{E} = -\vec{\nabla}\phi$, this means $\vec{E} = 0$ inside.



Electrostatic Energy
$$U_E[\rho] = \frac{1}{8\pi\epsilon_0} \int_V dV \int_V dV' \frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \int_V dV\, \rho(\vec{r})\phi_{ext}(\vec{r})$$

We have to find ϕ inside V such that $U_E(\rho)$ is a minimum.

We have the constraint that $\boxed{Q = \int_V dV\, \rho(\vec{r})}$ is a constant.

By the method of Lagrange multipliers, we can minimize unconstrained, this:

$$I[\rho] = U_E[\rho] - \lambda Q$$

So, to first order, we insist:

$$\delta I = I[\rho + \delta\rho] - I[\rho] = 0$$

To calculate δI, we let $\rho(\vec{r}) \longrightarrow \rho(\vec{r}) + \delta\rho(\vec{r})$

and $\rho(\vec{r}') \longrightarrow \rho(\vec{r}') + \delta\rho(\vec{r}')$

Then,
$$I[\rho+\delta\rho] = \frac{1}{8\pi\epsilon_0} \int_V dV \int_{V'} dV' \frac{[\rho(\vec{r}) + \delta\rho(\vec{r})][\rho(\vec{r}') + \delta\rho(\vec{r}')]}{|\vec{r}-\vec{r}'|}$$
$$+ \int_V dV [\rho(\vec{r}) + \delta\rho(\vec{r})]\phi_{ext}(\vec{r}) - \lambda \int_V dV [\rho(\vec{r}) + \delta\rho(\vec{r})]$$

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~~$I[\rho+\delta\rho] - I[\rho]$ is:~~

~~Interchanging $\vec{r}$ and $\vec{r}'$, we get:~~

the same

$$I[\rho+\delta\rho] = \frac{1}{8\pi\epsilon_0}\int_V dV \int_{V'} dV' \, \frac{[\rho(\vec{r})\delta\rho(\vec{r}') + \rho(\vec{r}')\delta\rho(\vec{r}) + \rho(\vec{r})\rho(\vec{r}')]}{|\vec{r}-\vec{r}'|}$$

$$+ \int_V dV [\rho(\vec{r}) + \delta\rho(\vec{r})]\,\phi_{ext}(\vec{r}) - \lambda\int_V dV[\rho(\vec{r})+\delta\rho(\vec{r})]$$

So, $I[\rho+\delta\rho] - I[\rho]$ is:

$$\frac{1}{4\pi\epsilon_0}\int_V dV \int_{V'} dV' \left[\frac{\rho(\vec{r}')\,\delta\rho(\vec{r})}{|\vec{r}-\vec{r}'|}\right] + \int_V dV\, \delta\rho(\vec{r})\,\phi_{ext}(\vec{r})$$

$$-\lambda\int_V dV\, \delta\rho(\vec{r})$$

We insist this is zero to minimize U_E given constraint.

That is,

$$\delta I = \int_V dV\, \delta\rho(\vec{r})\left[\frac{1}{4\pi\epsilon_0}\int_V dV' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \phi_{ext}(\vec{r}) - \lambda\right] = 0$$

Since $\delta\rho(\vec{r})$ is arbitrary, the integrand must be zero:

That is:

$$\underbrace{\frac{1}{4\pi\epsilon_0}\int_V dV' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \phi_{ext}(\vec{r})}_{\text{So the potential energy of all points inside the potato!}} = \underbrace{\lambda}_{\text{constant}} \quad \text{for } \vec{r}\in V$$

Thus $\vec{E}$ is 0 inside.