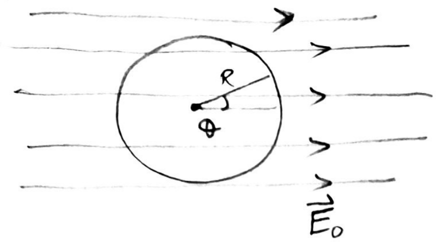
Example



Lecture 6

A conducting sphere sits in a uniform $\vec{E}$ field, $\vec{E}_0$
What is the dipole moment aquired by the sphere?

Solution

Inside the sphere, the charges must produce an electric field

$$\vec{E}_{self} = -\vec{E}_0$$

So, $\Phi_{self} = E_0 z = E_0 r\cos\theta$

Outside the sphere, it must be a generalized multipole expansion.

So, $$\Phi_{self}(r>R) = \frac{A}{r} + \frac{B}{r^2}\cos\theta + \frac{C}{r^3}\left(\frac{3\cos^2\theta - 1}{2}\right) + \cdots$$

Since potentials must match across any boundary of charge, only the $\frac{B}{r^2}\cos\theta$ survives!

Therefore, $$E_0 R\cos\theta \equiv \frac{B}{R^2}\cos\theta$$

$$\therefore\ B = E_0 R^3$$

So, outside the sphere, due only to the sphere,

$$\Phi_{self}(r>R) = \frac{E_0 R^3}{r^2}\cos\theta$$

Now, compare this to our result for the potential due to a charged sphere with $\sigma = \sigma_0\cos\theta$:

$$\Phi(\vec{r}) = \frac{1}{3}\frac{\sigma_0}{\epsilon_0}\frac{R^3}{r^2}\cos\theta$$

So, $\boxed{\sigma_0 = 3\epsilon_0 E_0}$ ← the scale of the induced charge is exactly proportional to the $\vec{E}_0$!

Let us go a little farther. What is the induced dipole moment of the sphere?

We know from Lecture 3, $p_z = \frac{4\pi}{3}\sigma_0 R^3$

So, $p_z = 4\pi R^3 \epsilon_0 E_0$

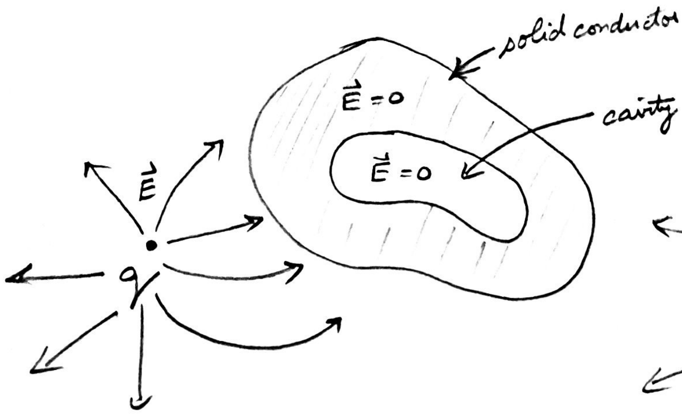
That is, $\vec{p} = \underbrace{4\pi R^3 \epsilon_0}_{\text{Polarizability of the material.}} \vec{E}_0$

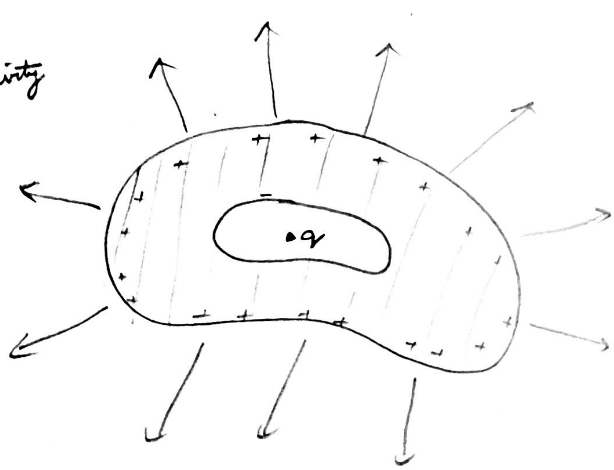
It is relationship between applied $\vec{E}$ field and induced dipole moment $\vec{p}$.

Represented by uppercase "P".

For a compact, polarizeable object, $P \propto$ Volume.

Screening and Shielding





- Any point inside a cavity in a conducting body is perfectly shielded from the effect of any charges outside.

- A charge placed on the inside of a cavity in a conductor expresses itself outside. $\oint_S \vec{E}\cdot d\vec{s} = q/\epsilon_0$ always!

Capacitance

It is easiest to think first in terms of two conductors with opposite charge. The ratio of the charge to the voltage difference between them is the capacitance. It is a purely geometric quantity.

Being a conductor, A must have exactly one potential... V_1

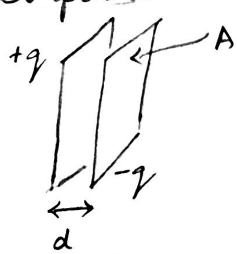
" " " B must also have one potential V_2

The potential difference $V = V_1 - V_2$

Definition: $\boxed{C = \frac{q}{V}}$

Example 1

Parallel plates



Field between them ~ $E = \frac{\sigma}{\epsilon_0} = \frac{q}{A\epsilon_0}$

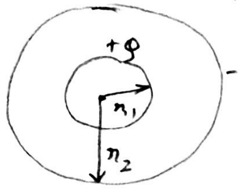
So, the potential difference: $V = \int_0^d E \cdot dr = Ed$

$\therefore V = \frac{qd}{A\epsilon_0}$

So, $\boxed{C = \frac{q}{V} = \frac{A\epsilon_0}{d}}$

C always has dimensionality of $\epsilon_0 \times$ length.

Example 2 Two concentric spherical conducting shells.



$$E = \frac{q}{4\pi\epsilon_0 r^2}$$ between shells 1 and 2.

So,
$$V = \int_{r_1}^{r_2} \frac{q}{4\pi\epsilon_0 r^2}\, dr$$

$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

So, $$\boxed{C = \frac{q}{V} = 4\pi\epsilon_0\left[\frac{r_1 r_2}{r_1 - r_2}\right]}$$

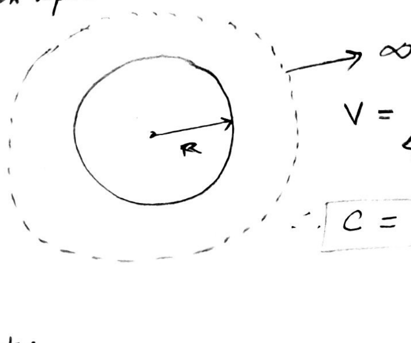
Exercise

Please work out by yourself the C of two concentric cylidrical shells.

Self Capacitance

Where the other charged conductor is at ∞ with $V = 0$.

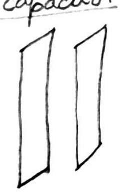
- What is it for a thin spherical shell?



$$V = \frac{q}{4\pi\epsilon_0 R}$$

$$\therefore \boxed{C = 4\pi\epsilon_0 R}$$

Energy stored in a capacitor



Let us start with two neutral parallel plates, and begin to move electrons from one to the other. You begin to do work against a field as you go.

$dW = V dq$, where V is the potential difference at a moment during the move.

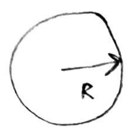
$$\therefore dW = \frac{q}{C}dq$$

$$\therefore W = \int_0^q \left(\frac{q}{C}\right) dq = \frac{q^2}{2C}$$

So, energy stored in the capacitor is

$$\boxed{U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C V^2}$$

What is the energy of a single shell of charge Q?



$$C = 4\pi\epsilon_0 R$$

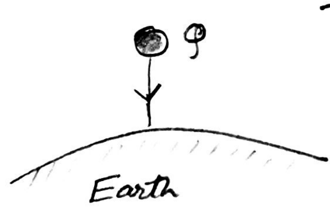
So, $U = \frac{1}{2}\frac{Q^2}{C} = \frac{Q^2}{8\pi\epsilon_0 R}$

Have we seen this before?

Yes! In Lecture 2 where we calculated the self energy of a thin shell..

It is also exactly equal to the integral of $\frac{\epsilon_0}{2} E^2$ outside the shell.

What does it mean to ground a conductor?



The Earth is a huge, relatively infinite C.

Because $C = 4\pi\epsilon_0 R$.

So, any charge flowing to or from it will hardly change the potential of the Earth.

$$U_{total} = \frac{Q_1^2}{C_{potato}} + \frac{Q_2^2}{C_{Earth}}$$

Because $C_{Earth} \gg C_{potato}$,

U_{total} is minimum when Q_1 moves to the Earth.

So, that's what happens!

Conversely, one can pull charge the Earth easily as well if an additional charge nearby, floating, induces charge in the conductor.