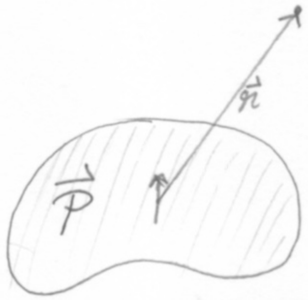
Lecture 8 Dielectrics

- Field produced by a polarized object.
  Not the field that may have produced the $\vec{P}$, but what $\vec{P}$ produces.



Since polarized material can be considered as composed of lots of small dipoles.

$$\int_V dV\, \vec{P} = \vec{p}$$

So, the potential $\phi(\vec{r})$ at a distance from this object is

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0} \int_V dV\, \frac{\vec{P}(\vec{r}_s)\cdot(\vec{r}-\vec{r}_s)}{|\vec{r}-\vec{r}_s|^{3}}$$

Now, consider that $\vec{\nabla}_{r_s}\left(\frac{1}{|\vec{r}-\vec{r}_s|}\right) = \frac{\vec{r}-\vec{r}_s}{|\vec{r}-\vec{r}_s|^3}$

So, $$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V dV\, \vec{P}(\vec{r}_s)\cdot\vec{\nabla}_{r_s}\left(\frac{1}{|\vec{r}-\vec{r}_s|}\right)$$

We use $$\vec{\nabla}_s\cdot\left(\frac{\vec{P}}{|\vec{r}-\vec{r}_s|}\right) = \vec{P}\cdot\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}_s|}\right) + \frac{\vec{\nabla}\cdot\vec{P}}{|\vec{r}-\vec{r}_s|}$$

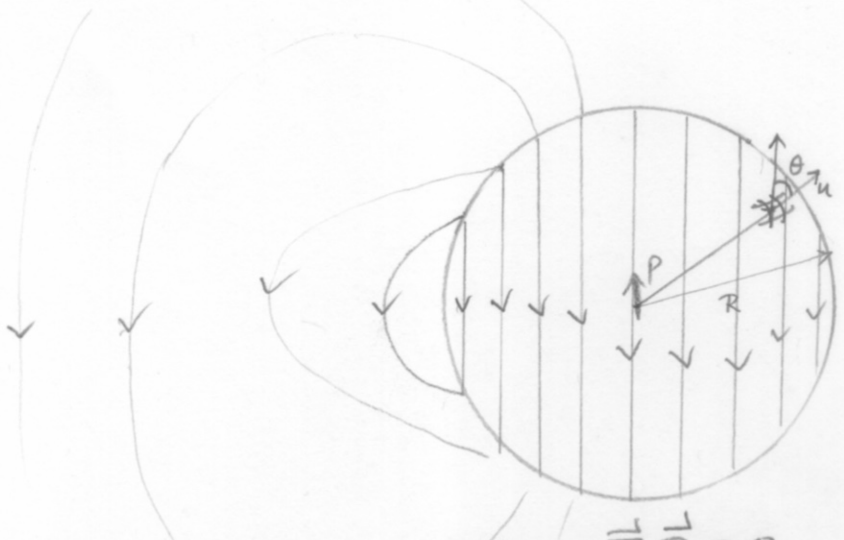
$$\therefore\ \phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\int_V dV\, \vec{\nabla}_s\cdot\left(\frac{\vec{P}}{|\vec{r}-\vec{r}_s|}\right) - \int_V dV\, \frac{\vec{\nabla}\cdot\vec{P}}{|\vec{r}-\vec{r}_s|}\right]$$

So, $$\phi(\vec{r}) = \underbrace{\frac{1}{4\pi\epsilon_0}\oint_S dA\, \frac{\vec{P}\cdot\hat{n}}{|\vec{r}-\vec{r}_s|}}_{\text{Surface charge}} + \underbrace{\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho\, dV}{|\vec{r}-\vec{r}_s|}}_{\text{Volume charge}}$$

where, $\boxed{\sigma_p = \vec{P}\cdot\hat{n}}$ & $\boxed{\rho_p = -\vec{\nabla}\cdot\vec{P}}$

So, we can calculate the charges first and find ϕ, $\vec{E}$ from them.

Find the field produced by a uniformly polarized sphere



The volume charge density $= \vec{\nabla} \cdot \vec{P} = 0$

The surface charge density $= \vec{P} \cdot \hat{\eta} = P\cos\theta$

We have seen this before, so,

$$\phi(r,\theta) = \begin{cases} \dfrac{P}{3\epsilon_0} r\cos\theta & ; \text{ inside the sphere} \\ \dfrac{P}{3\epsilon_0}\dfrac{R^3}{r^2}\cos\theta & ; \text{ outside the sphere} \end{cases}$$

[As a physicist, learn to think dimensionally.]

Since $\vec{E} = -\vec{\nabla}\phi$, the field inside is uniform

$$\boxed{\vec{E}_{int} = \frac{-P}{3\epsilon_0}\hat{z}}$$ because $r\cos\theta = z$

And outside, it is dipolar.

So, the total field inside the object is the sum of the external electric field and this electric field

So, if $\vec{P} = \alpha \vec{E}_{ext}$

Then, $\vec{E} = \vec{E}_{ext} + \vec{E}_{int}$

$$= \vec{E}_{ext} - \frac{P\hat{z}}{3\epsilon_0}$$

$$= \vec{E}_{ext} - \alpha\frac{\vec{E}_{ext}}{3\epsilon_0}$$

$$\vec{E} = \vec{E}_{ext}\left(1 - \frac{\alpha}{3\epsilon_0}\right)$$

an attenuated version of the $\vec{E}$ field.

What does Gauss' Law look like inside a dielectric?

Field due to polarization of the medium = field due to bound charge.

$$\rho = \rho_p + \rho_f$$

$\rho_f \longrightarrow$ charges that induced the polarization. Free charges in the dielectric.

So, $$\vec{\nabla}\cdot\vec{E} = \frac{\rho_p + \rho_f}{\epsilon_0}$$

$$= \frac{-\vec{\nabla}\cdot\vec{P} + \rho_f}{\epsilon_0}$$

$E(1+\chi_0) = \frac{\rho}{\epsilon_0}$

$E = \frac{\rho}{\epsilon}$

So; $$\vec{\nabla}\cdot\underbrace{(\epsilon_0\vec{E} + \vec{P})}_{\text{Electric displacement}} = \rho_f$$

So, $$\boxed{\vec{\nabla}\cdot\vec{D} = \rho_f} \quad \text{or} \quad \oint \vec{D}\cdot d\vec{A} = q_{free}$$

One could be tempted to think ρ_{free} is the only source of D.

But, $\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P}$ which may not be 0.

So far, results have been general

But, what if the $\vec{P}$ is proportional, linearly, to $\vec{E}$?

Then, $\vec{P} = \epsilon_0 \chi_0 \vec{E}$

$\chi_0 \longleftarrow$ electric susceptibility (very related to the polarizability α)

susceptibility $\rightarrow$ dimensionless

Then $$\vec{D} = \epsilon_0\vec{E} + \vec{P}$$

$$\vec{D} = \epsilon_0(1+\chi_e)\vec{E}$$

or, $$\boxed{\vec{D} = \epsilon\vec{E}}$$

$\epsilon \longrightarrow$ permittivity of the dielectric

$$\boxed{\epsilon = \epsilon_0(\underbrace{1+\chi_e}_{\text{dielectric constant} = K})}$$

dielectric constant $= K$

1 for vacuum

>1 for materials

∴ How does the $\vec{E}$ & ϕ vary when placed inside a <u>linear</u> dielectric?

$$\vec{\nabla}\cdot\vec{D} = \rho_{free}$$ ← this is what is observable, macroscopically

And $\vec{D} = \epsilon\vec{E}$ for <u>a linear</u> dielectric

$\epsilon = \epsilon_0 K$

So, $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon$,,

$$\vec{E} = \frac{q}{4\pi\epsilon}\frac{\hat{r}}{r^2}$$

$E = \frac{\epsilon_0}{\epsilon}E$

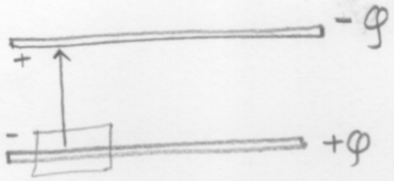
$E = \frac{E_0}{K}$

$$\phi = \frac{q}{4\pi\epsilon r}$$

$\frac{1}{2}QV$

Parallel plate capacitor with fixed charge

$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$



$Q = CV$

$$E = \frac{Q}{A\epsilon}$$

$$V = \int E\cdot dz = \frac{Qd}{A\epsilon}$$

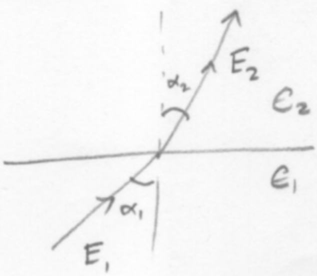
This implies dielectric will be dragged in for fixed Q.

So, $\boxed{C = \frac{A\epsilon}{d}}$ → since $\epsilon = \epsilon_0 K$ this is higher.

What if you have a nonlinear dielectric?

$$\vec{P} = \epsilon_0\chi_0\vec{E} + \epsilon_0\chi_1 E^2\hat{E}$$

Refraction of $\vec{E}$ field at interface of two dielectrics



$$E_1\sin\alpha_1 = E_2\sin\alpha_2$$

$$E_1\epsilon_1\cos\alpha_1 = E_2\epsilon_2\cos\alpha_2$$

So, $$\frac{\tan\alpha_1}{\epsilon_1} = \frac{\tan\alpha_2}{\epsilon_2}$$