Lecture 9

Laplace's Equation

$$\nabla^2 \phi = 0$$; is valid when some space has no charge.

In 1-D

$$\frac{d^2\phi}{dx^2} = 0$$

General solution is $\phi(x) = mx + c$

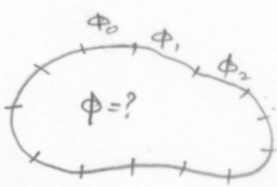
Fixed by boundary conditions.

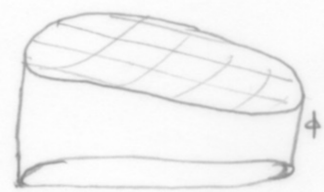
Almost trivial observation: $\phi(x) = \frac{1}{2}\left[\phi(x+a) + \phi(x-a)\right]$

This seemingly trivial observation translates to profound insight in 2 and 3 D.

In 2D

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$





Here $\phi(x,y) = \frac{1}{2\pi R} \oint \phi \, dl$ around (x,y)

Method of relaxation lends itself to computer simulations

In 3D

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

Does our insight still hold?

$$\Phi_{avg}(R) = \frac{1}{4\pi R^2} \oint \Phi(\vec{r}) \, dA$$

Now, $dA = R^2 \sin\theta \, d\theta \, d\phi$

So, $$\Phi_{avg}(R) = \frac{1}{4\pi R^2} \oint \Phi(R,\theta,\phi) \, R^2 \sin\theta \, d\theta \, d\phi$$

So, $$\frac{d\Phi_{avg}}{dR} = \frac{1}{4\pi} \oint \frac{\partial\Phi}{\partial R}(R,\theta,\phi) \sin\theta \, d\theta \, d\phi$$

Now, $$\frac{\partial\Phi(R,\theta,\phi)}{\partial R} = \vec{\nabla}\Phi \cdot \hat{r}$$

So,
$$\frac{d\Phi_{avg}}{dR} = \frac{1}{4\pi}\oint (\vec{\nabla}\Phi)\cdot\hat{r}\ \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{4\pi R^2}\oint (\nabla\Phi)_{\hat{r}}\ R^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{4\pi R^2}\oint (\vec{\nabla}\Phi)_{\hat{r}}\cdot d\vec{A}$$

$$= \frac{1}{4\pi R^2}\int_V \nabla^2\Phi\, dV$$

But, $\nabla^2\phi = 0$; $\vec{\nabla}\cdot\vec{E} = 0$ for a charge-less region

So, $$\frac{d\Phi_{avg}}{dR} = 0$$

This means Φ_{avg} has no~~t~~ dependence on R

So, $\Phi_{avg}(R=0) = \Phi_{avg}(R)$